From the notion of a morphism between QFTs to SymTFT

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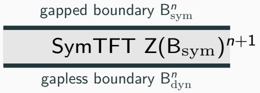
References: K.-Wen-Zheng:1502.01690, 1702.00673, K.-Zheng:1705.01087,1905.04924,1912.01760

The proposal of sandwich construction or SymTFT or topological holography (within high energy community): Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Freed-Moore-Teleman:2209.07471, Apruzzi-Bonetti-Etxebarria-Hosseini-Schäfer-Nameki:2112.02092

$$\mathsf{X}^n := \mathsf{B}^n_{\mathrm{sym}} \boxtimes_{\mathsf{C}^{n+1}} \mathsf{B}^n_{\mathrm{dyn}} := \Big\{$$



1. gapped boundary $\mathsf{B}^n_{\mathrm{sym}}$ encodes the information of non-invertible symmetries;
2. gapless boundary $\mathsf{B}^n_{\mathrm{dyn}}$ encodes the information of dynamical data.

In this talk, I will show that this idea of SymTFT is a consequence of the mathematical theory of gapped/gapless boundaries of 2+1D topological orders developed in K.-Zheng:1705.01087 (more details in 1905.04924,1912.01760). This theory provides a precise meaning of above sandwich expansion and a powerful way to compute the sandwich.

A morphism of QFT and boundary-bulk relation

Question: How to define a morphism between two QFT's?

A well known answer: A usual definition of a morphism between two QFT's is a domain wall between two QFT's. A domain wall provides a physical realization of the mathematical notion of a bimodule because the domain wall is naturally equipped with the two-side action of the operators in two QFT's.

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However, such a definition of a morphism (as a bimodule) is less fundamental because it does not preserve the algebraic structures of a QFT. As a consequence, such a definition of a morphism (as a bimodule) only distinguishe QFT's up to Morita equivalences. We want a morphism preserving the algebraic structures of a QFT.

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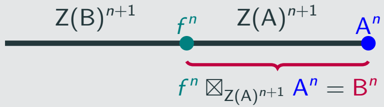
In this talk, we will only discuss the notion of a morphism for a very special QFT, called (potentially anomalous) topological order (i.e., a gapped liquid phase at zero temperature without usual symmetry). Mathematically, an n+1D anomaly-free topological order should be roughly viewed as an n+1D fully dualizable TQFT defined on a spatial open n-disk.

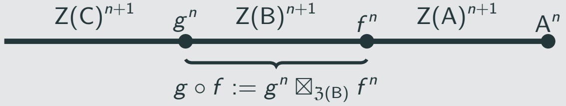
An anomalous nD topological order A^n is a gapped boundary of a non-trivial anomaly-free topological order $Z(A)^{n+1}$, which is called the bulk of A^n or the gravitation anomaly of A^n. K.-Wen:1405.5858

Definition (K.-Wen-Zheng:1502.01690, 1702.00673)

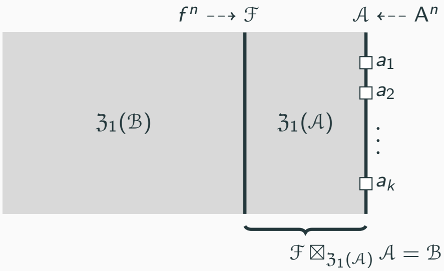
A morphism $f : A^n \to B^n$ between two topological orders A^n and B^n (both having gapped bulks) is a gapped domain wall f^n between $Z(A)^{n+1}$ and $Z(B)^{n+1}$ such that



The composition of two morphisms $f : A^n \to B^n$ and $g : B^n \to C^n$ is defined as follows:



This definition of a morphism
coincides with the mathematical notion of
a monoidal functor between two fusion n-categories



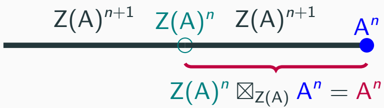
$$\{\text{monoidal functors } \mathcal{A} \to \mathcal{B}\} \quad \overset{1:1}{\Longleftrightarrow} \quad \{\text{“domain walls” between } \mathfrak{Z}_1(\mathcal{B}) \text{ and } \mathfrak{Z}_1(\mathcal{A}) \text{ s.t. } \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} = \mathcal{B}\}$$

$$(\mathcal{A} \xrightarrow{f} \mathcal{B}) \quad \longmapsto \quad \mathcal{F} := \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}({}_f\mathcal{B}, {}_f\mathcal{B}),$$

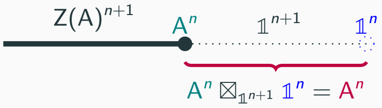
$$\left(\mathcal{A} \xrightarrow{f} \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} = \mathcal{B}\right) \quad \longleftarrow \quad \mathcal{F}$$

These “domain walls” are “closed monoidal $\mathfrak{Z}_1(\mathcal{B})$-$\mathfrak{Z}_1(\mathcal{A})$-bimodules” in mathematics. These two constructions are inverse of each other. K.-Zheng:1507.00503, 2107.03858

Examples 1: $A^n \xrightarrow{\mathrm{id}_A} A^n$ is defined by the trivial domain wall $Z(A)^n$ in $Z(A)^{n+1}$.



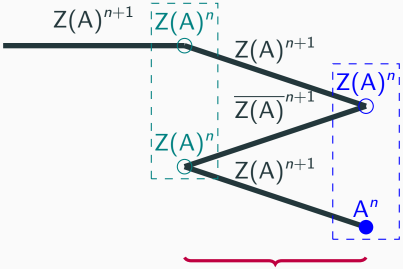
Examples 2: Let $\mathbb{1}^n$ be the trivial nD topological order. There is a canonical morphism $\iota : \mathbb{1}^n \to A^n$ defined by:



Examples 3: There is a canonical morphism

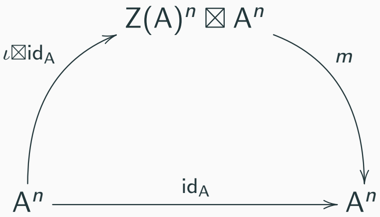
$$Z(A)^n \boxtimes A^n \xrightarrow{m} A^n$$

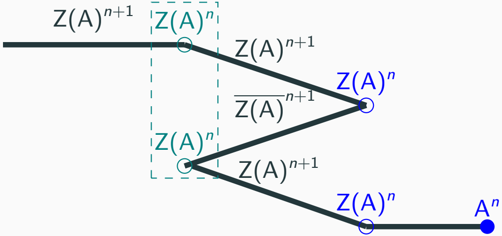
defined as follows



$$(Z(A)^n \boxtimes Z(A)^n) \boxtimes_{Z(A)\boxtimes\overline{Z(A)}\boxtimes Z(A)} (Z(A)^n \boxtimes A^n) = A^n$$

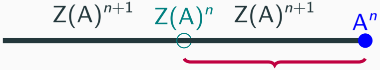
These three morphisms: id_A, ι, m form a commutative diagram.





$$(Z(A)^n \boxtimes Z(A)^n) \boxtimes_{Z(A)\boxtimes\overline{Z(A)}\boxtimes Z(A)} (Z(A)^n \boxtimes A^n) = A^n$$

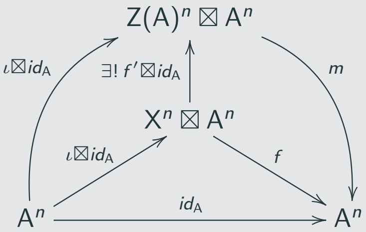
$=$



$$Z(A)^n \boxtimes_{Z(A)} A^n = A^n$$

Theorem (K.-Wen-Zheng:1502.01690, 1702.00673)

The pair $(Z(A)^n, m)$ satisfies the universal property of center. That is, if X^n is an nD topological order equipped with a morphism $f : X^n \boxtimes A^n \to A^n$ rendering the same triangle commutative, then there is a unique morphism $f' : X^n \to Z(A)^n$ such that the following diagram is commutative:



Remark: This theorem simply says that the *bulk* $Z(A)^n$ is the center of the boundary A^n.

This universal property works for all the well-known notions of centers.

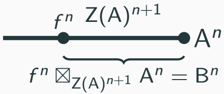
1. When A is a group and m is a group homomorphism, it recovers the center of a group $\mathfrak{Z}(\mathrm{A}) = \{z \in \mathrm{A} | zg = gz, \forall g \in \mathrm{A}\}$.
2. When A is an algebra and m is an algebraic homomorphism, it recovers the usual center of an algebra $\mathfrak{Z}(\mathrm{A}) = \{z \in \mathrm{A} | za = az, \forall a \in \mathrm{A}\}$.
3. When A is a fusion category and m is a monoidal functor, it recovers the Drinfeld center.
4. When A is a braided fusion category and m is a braided monoidal functor, it recovers the Müger center.
5. The center of open-string CFT is the closed CFT and is called a full center. Fjelstad-Fuchs-Runkel-Schweigert:math.CT/0512076, K.-Runkel:0708.1897, Davydov:0908.1250
6. Generalized Deligne Conjecture (Kontsevich): the E_n-center of an E_n-algebra is an E_{n+1}-algebra. Lurie's book "Higher Algebras"

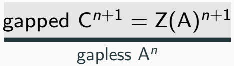
Gapless boundaries of 2+1D topological orders

Consequences of boundary-bulk relation:

1. It leads to the classification of all $n+1$D anomaly-free topological orders (up to invertible ones) by fusion n-categories with a trivial E_1-center, or equivalently, by braided fusion n-categories with a trivial E_2-center; K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, Johnson-Freyd:2003.06663, K.-Zheng:2011.02859,2107.03858

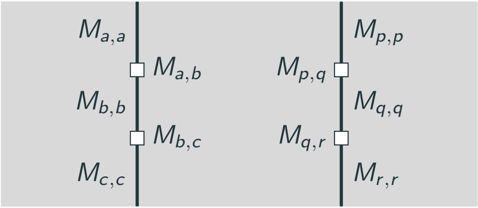
2. The proof applies to an $n+1$D topological order C^{n+1} with a gapless boundary A^n.





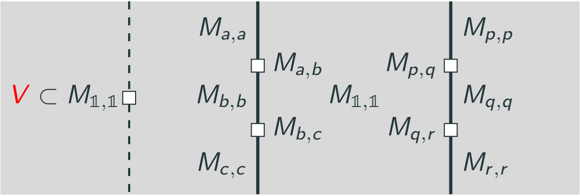
Since we already have a precise categorical description of C^{n+1} as a braided fusion n-categories $\mathcal{C}$ with a trivial E_2-center, then we can find the categorical description of a gapless boundary A^n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$. Since $\mathfrak{Z}_1(\mathcal{C}) \simeq \mathcal{C} \boxtimes \overline{\mathcal{C}}$, we can view $? \simeq \sqrt{\mathcal{C}}$.

Consider the 1+1D worldsheet of a gapless boundary of the 2+1D topological order $(\mathcal{C}, c)$.



It turns out that the macroscopic observables on the 1+1D worldsheet form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ K.-Zheng:1705.01087, 1905.04924, 2011.02859, K.-Wen-Zheng:2108.08835, where

1. $a, b, c \in \mathcal{S}$ are the labels of topological defect lines (TDL);
2. $M_{a,b}$ is the spaces of fields living on the 0D defect junction; in particular, it means that the space of fields living on the TDL label by 'a' is just $M_{a,a}$;
3. OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ of defect fields defines a kind of 'composition map' such that all $\{M_{a,b}\}_{a,b \in \mathcal{S}}$ together form a structure similar to that of a category $(\{\hom(a,b)\}_{a,b \in \mathcal{C}})$.
4. We will show next that $M_{a,b}, \circ \in \mathcal{B}$ and we obtain a $\mathcal{B}$-enriched category ${}^{\mathcal{B}}\mathcal{S}$.



Let $\mathbb{1} \in \mathcal{S}$ be the label of the trivial TDL. Then fields in $M_{\mathbb{1},\mathbb{1}}$ can live in the 2-cell. A subalgebra of fields generated by $\langle T(z,\bar{z})\rangle \subset M_{\mathbb{1},\mathbb{1}}$ is transparent to all TDL's. Without loss of generality, we assume that $\langle T(z,\bar{z})\rangle \subset V \subset M_{\mathbb{1},\mathbb{1}}$ is transparent to all TDL's. Assume V is rational, i.e. Mod_V is a MTC. Moore-Seiberg:1989, Huang:math/0502533, K.:math/0603065

$M_{a,b}$ is clearly a V-module (with a 2-dimensional V-action), i.e. $M_{a,b} \in \mathrm{Mod}_V$. The compatibility between the OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ and the V-action is equivalent to a morphism $M_{b,c} \otimes_V M_{a,b} \xrightarrow{\circ} M_{a,c}$ in Mod_V. As a consequence, we obtain an Mod_V-enriched fusion category ${}^{\mathrm{Mod}_V}\mathcal{S}$, where $\hom_{{}^{\mathrm{Mod}_V}\mathcal{S}}(a,b) = M_{a,b}$ and $\otimes$ is the horizontal fusion of TDLs. It turns out that all correlation functions and the OPE among defect fields can be recovered from $(V, {}^{\mathcal{B}}\mathcal{S})$ for $\mathcal{B} = \mathrm{Mod}_V$.Huang:math/0303049,math/0502533, Fuchs-Runkel-Schweigert:2001-2006, Huang-Kirillov-Lepowsky:1406.3420, K.:math/0612255, K-.Runkel:0807.3356, Davydov-K.-Runkel:1307.5956

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

The 'rational' gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, \mathcal{S})$ or $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where

1. *for a chiral gapless boundary, V is a rational VOA of central charge c;* Huang:math/0502533
   for a non-chiral gapless boundary, V is a rational full field algebra $(c_L - c_R = c)$ K.-Huang:math/0511328;
   for a gapped boundary, $V = \mathbb{C}$. Kitaev-K.:1104.5047
2. *$\mathcal{S}$ is a fusion category equipped with a braided equivalence $\phi : \mathcal{C} \boxtimes \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathfrak{Z}_1(\mathcal{S})$.*

Moreover, the restriction $\phi : \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})} = \overline{\mathrm{Mod}_V}$ is a braided equivalence, which determines a $\mathcal{B}$-action on $\mathcal{S}$ and a $\mathcal{B}$-enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction, i.e. $M_{a,b} = [a, b] \in \mathcal{B}$.
Lindner:1981,Morrison-Penneys:1701.00567

Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

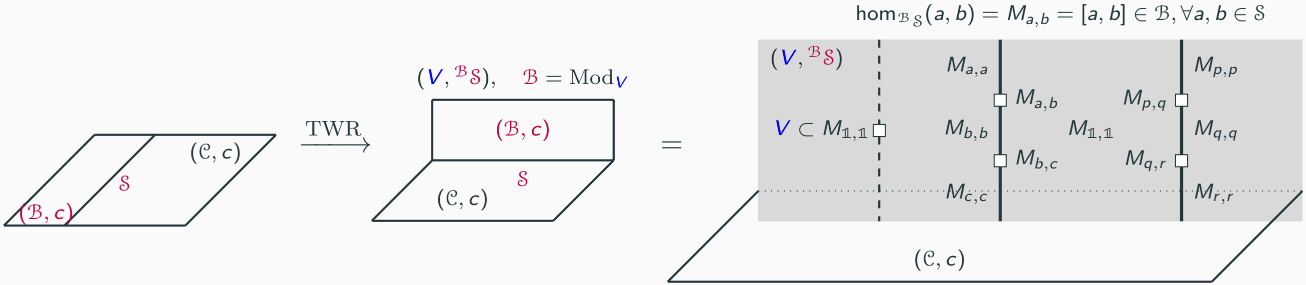
The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a pair* $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$, *where*

1. $\mathsf{X}_{\mathrm{lqs}} = (V, \phi)$ *is called local quantum symmetry (containing dynamical information);*
2. $\mathsf{X}_{\mathrm{top}} = {}^{\mathcal{B}}\mathcal{S}$ *is called topological skeleton (recall* $\mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})}$*).*

Moreover, $\mathsf{X}_{\mathrm{top}}$ *can be obtained by topological Wick rotation and* $\mathfrak{Z}_1(\mathsf{X}_{\mathrm{top}}) \simeq \mathcal{C}$.



When $(\mathcal{C}, c)$ is trivial, i.e., $\mathcal{B} \simeq \mathfrak{Z}_1(\mathcal{S})$, we obtain a holographic duality between a 3D theory and a 2D theory.

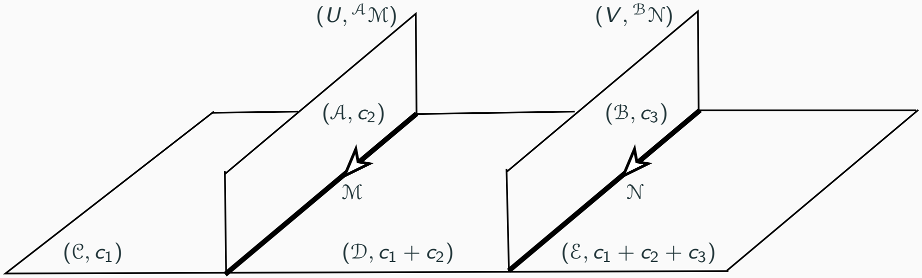
Examples of gapped and gapless boundaries:

1. For a non-chiral 2+1D topological order $(\mathfrak{Z}(\mathcal{A}), 0)$, where $\mathcal{A}$ is a fusion category, the pair $(\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{A} = \mathcal{A})$, where $\mathrm{Mod}_{\mathbb{C}} = \mathrm{Vec}$, defines a gapped boundary.
2. For the E_8 invertible 2+1D topological order $(\mathcal{C}, c) = (\mathrm{Vec}, 8)$, the pair $(V_{E_8}, {}^{\mathrm{Vec}}\mathrm{Vec})$, where $\mathrm{Mod}_{V_{E_8}} = \mathrm{Vec}$, defines a non-trivial gapless boundary.
3. Let V be a rational VOA and $\mathcal{C} = \mathrm{Mod}_V$ is MTC.
   - The pair $(V, {}^{\mathcal{C}}\mathcal{C})$ defines a gapless boundary of $(\mathcal{C}, c)$ and $\mathfrak{Z}({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$.
   - The triple $(V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C})$ defines an anomaly-free 1+1D CFT, where $\mathrm{Mod}_{V\otimes_{\mathbb{C}}\overline{V}} = \mathcal{C} \boxtimes \overline{\mathcal{C}}$, and

$$M_{\mathbb{1}_\mathcal{C}, \mathbb{1}_\mathcal{C}} = [\mathbb{1}_\mathcal{C}, \mathbb{1}_\mathcal{C}] = \oplus_{i\in\mathrm{Irr}(\mathcal{C})}\, i \boxtimes i^*;$$
$$M_{a,b} = [a, b] = \oplus_{i\in\mathrm{Irr}(\mathcal{C})}\, (i \otimes b \otimes a^*) \boxtimes i^*$$

   Moreover, we have $\mathfrak{Z}_1({}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{C}) = \mathrm{Vec}$.

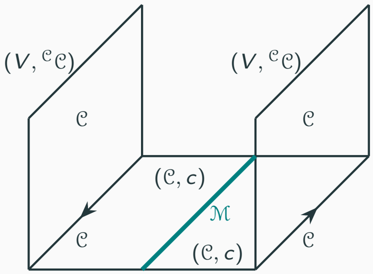
How to compute the fusion of two gapless domain walls? Let U, V be VOAs or full field algebras.



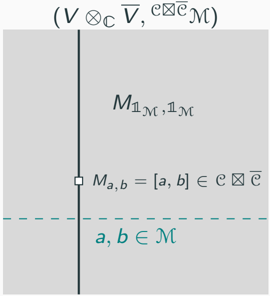
K.-Zheng:1705.01087

$$(U, {}^{\mathcal{A}}\mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, {}^{\mathcal{B}}\mathcal{N}) = (U \otimes_{\mathbb{C}} V, {}^{\mathcal{A}\boxtimes\mathcal{B}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

Consider a 2+1D topological order $(\mathcal{C}, c)$ with the canonical gapless boundary $(V, {}^{\mathcal{C}}\mathcal{C})$, where V is a VOA and $\mathcal{C} = \mathrm{Mod}_V$, and a gapped domain wall $\mathcal{M}$: K.-Zheng:1705.01087



dimensional reduction
to a 1+1D RCFT

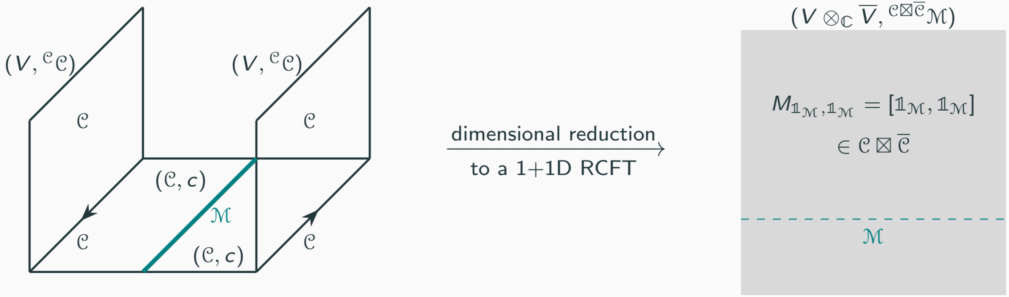


$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}).$$

$M_{\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}} = [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ recovers the modular invariant 1+1D rational CFT canonically associated with $\mathcal{M}$. This establishes a one-to-one correspondence:

$$\{\text{gapped domain walls in the 2+1D topological order } (\mathcal{C}, c)\} \xleftrightarrow{1:1} \{\text{modular invariant CFT's extending } V \otimes \overline{V}\}$$

$$\mathcal{M} \xmapsto{1:1} [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$$



$\{$modular invariant CFT's extending $V \otimes_{\mathbb{C}} \overline{V}\}$

$\overset{1:1}{\longleftrightarrow} \{$Lagrangian algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}\}$ K.-Runkel:0807.3356, Müger:0909.2537

$\overset{1:1}{\longleftrightarrow} \{$indecomposable $\mathcal{C}$-module categories$\}$ K.-Runkel:0708.1897, Davydov-Müger-Nikshych-Ostrik:1009.2117

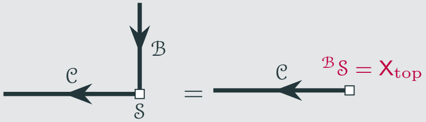
$\overset{1:1}{\longleftrightarrow} \{$gapped boundaries $\mathcal{M}$ of $(\mathcal{C} \boxtimes \overline{\mathcal{C}}, 0)\}$ Kitaev-K.:1104.5047, K.:1307.8244

$\overset{1:1}{\longleftrightarrow} \{$gapped domain walls $\mathcal{M}$ in $(\mathcal{C}, c)\}$ folding trick

Topological Wick rotation, categorical symmetry and SymTFT

Topological Wick rotation in all dimensions: K.-Zheng:1905.04924,1912.01760,2011.02859

For a (potentially anomalous) quantum liquid $X = (X_{lqs}, X_{top})$, its topological skeleton X_{top} can be obtained by topological Wick rotation.



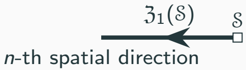
$\mathcal{S}$ is an "anomalous gapped boundaries" of an n+1D topological order $\mathcal{C}$

*n*D potentially gapless boundaries of an n+1D topological order $\mathcal{C}$

The boundary-bulk relation holds, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1(^{\mathcal{B}}\mathcal{S})$ K.-Zheng:in preparation.

Remark: In many examples, the same X_{top} can be realized by both gapped and gapless phases (depending on X_{lqs}). A mathematical theory of X_{lqs}, based on a theory of topological nets in *n*D generalizing that of conformal nets in 2D, was proposed and developed in K.-Zheng:2201.05726.

A new type of holographic dualities based on the idea of Topological Wick Rotation K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859: (nD is the spacetime dimension.)

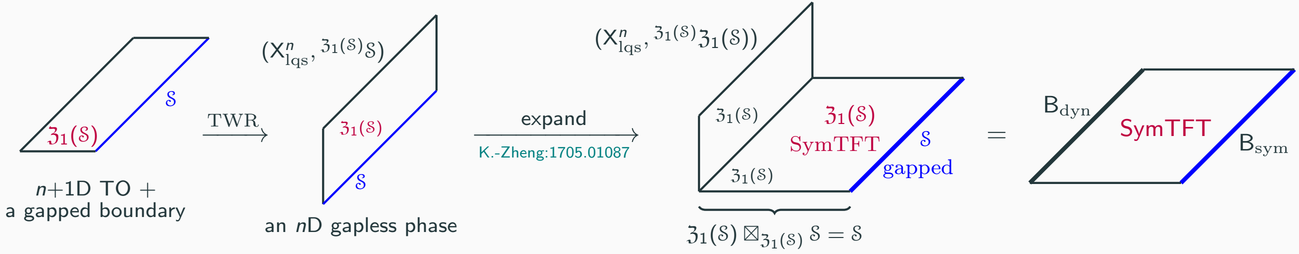


an n+1D topological order with a gapped boundary	an nD quantum liquid (SPT/SET/SSB/gapless) with an internal symmetry of finite type
$\mathcal{S}$ is the category of topological defects on the boundary	$\mathcal{S}$ is the category of topological defects { the superselection (charge) sectors of states }
$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological defects in the bulk	$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological sectors of operators { the spaces of non-local operators invariant under LOA}
$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$	$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$

This holographic dualities were explicitly/implicitly discovered and further studied by different groups of people in different contexts. An incomplete list:

- Topological Wick Rotation: K.-Zheng:1705.01087, 1905.04924, 1912.01760, K.-Zheng:2011.02859, K-Wen-Zheng:2108.08835, Xu-Zhang:2205.09656, Lu-Yang:2208.01572
- Categorical Symmetries or SymTO: Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178, Albert-Aasen-Xu-Ji-Alicea-Preskill:2111.12096, Chatterjee-Wen:2203.03596,2205.06244, Liu-Ji:2208.09101, Chatterjee-Ji-Wen:2212.14432
- 2D Statistical Models: Aasen-Mong-Fendley:2008.08598
- SymTFT and topological holography: Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Apruzzi-Bonetti-Etxebarria-Hosseini-Schafer-Nameki:2112.02092, Freed-Moore-Teleman:2209.07471, Apruzzi:2203.10063, Moradi-Moosavian-Tiwari:2207.10712, ...
- Other holographic phenomena: Strange correlators: Bal-Williamson-Vanhove-Bultinck-Haegeman-Verstraete:1801.05959; Generalized Kramers-Wannier dualities: Freed-Teleman:1806.00008, Lootens-Delcamp-Ortiz-Verstraete:2112.09091.

The idea of sandwich construction or SymTFT (i.e., expanding a gapless QFT X^n to a sandwich such that the fusion category symmetry $\mathcal{S}$ lives on one side of the sandwich and leave the dynamical data on the other side) can now be expressed precisely.



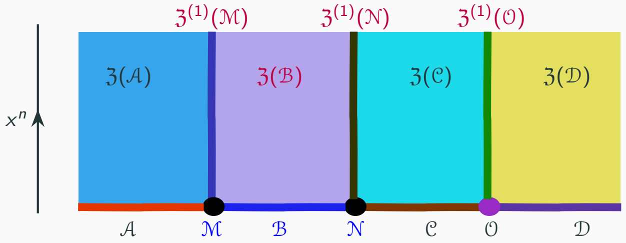
Sandwich expansions of a gapless QFT $X^n = (X^n_{\rm lqs}, {}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S})$ are not unique.

$$\{\text{sandwich expansions of } X^n\} \xleftrightarrow{1:1} \{f : \mathcal{T} \to \mathcal{S} \,|\, f \text{ is a monoidal functor; } \mathcal{T} \text{ is a fusion } n\text{-category}\}$$

The sandwich illustrated above is the one associated to the canonical functor $\mathcal{S} \xrightarrow{\mathrm{id}_{\mathcal{S}}} \mathcal{S}$.

Complete mathematical formulation of boundary-bulk relation:

$$\mathfrak{Z}^{(1)}(\mathcal{M}) \boxtimes_{\mathfrak{Z}(\mathcal{B})} \mathfrak{Z}^{(1)}(\mathcal{N}) = \mathfrak{Z}^{(1)}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N})$$



Theorem (Boundary-Bulk Relation of topological orders as the center functor)

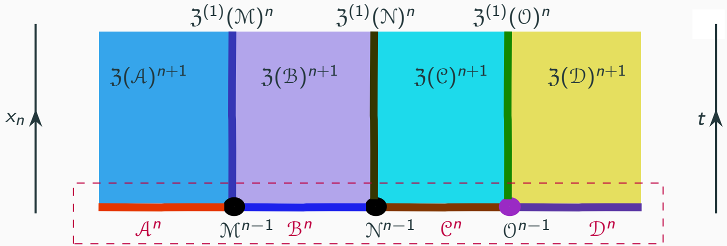
For fusion n-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and their bimodules ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{B}}, {}_{\mathcal{B}}\mathcal{N}_{\mathcal{C}}$, the following assignment

$$\mathcal{A} \mapsto \mathfrak{Z}(\mathcal{A}) \qquad \mathcal{M} \mapsto \mathfrak{Z}^{(1)}(\mathcal{M}) := \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{M}, \mathcal{M})$$

defines a functor (called the center functor), i.e.,

$$\mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{M}, \mathcal{M}) \boxtimes_{\mathfrak{Z}(\mathcal{B})} \mathrm{Fun}_{\mathcal{B}|\mathcal{C}}(\mathcal{N}, \mathcal{N}) \simeq \mathrm{Fun}_{\mathcal{A}|\mathcal{C}}(\mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}, \mathcal{M} \boxtimes_{\mathcal{B}} \mathcal{N}).$$

Topological Wick rotation is functorial.



We denote the category of the topological skeletons of nD anomaly-free quantum liquids by $\mathrm{QL}^n_{\mathrm{top}}$.

Theorem (K.-Zheng:2011.02859)

$$\mathrm{QL}^n_{\mathrm{top}} \simeq \bullet/(n+1)\mathrm{Vec},$$

where $(n+1)\mathrm{Vec} = \Sigma^n\mathrm{Vec} = \Sigma^{n+1}\mathbb{C}$ *was defined in Gaiotto-Johnson-Freyd:1905.09566.*

Conclusions

Main goal of this talk is to promote the notion of a morphism between QFT's, and to show that it is useful and powerful. Indeed, it alone had led us to the formal proof of boundary-bulk relation, to the discovery of topological Wick rotation, and to a unified mathematical theory of gapped and gapless quantum liquids, and to the study of the categories of quantum liquids.

Similar to the fact that category theory once revolutionized algebraic geometry by Grothendieck and his school, we believe that it will also revolutionize the theories of QFTs, phase transitions and perhaps quantum gravity. The important lesson is not only to borrow categorical language for physical use, but also to use the spirit of the category theory to ask new questions and find new truths. To define the notion of a morphism between QFT's is only an example of many possibilities.

Since the notion of a morphism is one of the most important concepts in mathematics, it is only reasonable that the notion of a morphism between QFT's is an important concept in physics.

Thank you !

Examples of non-chiral gapless edges: (skip unless people ask questions)

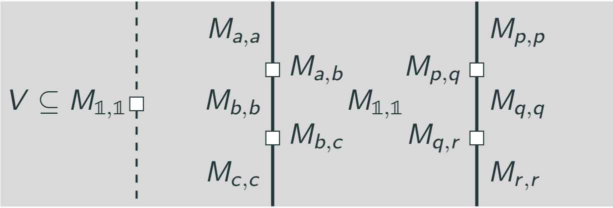
- Three modular tensor categories (MTC):
  1. $\mathrm{Is} := \mathrm{Mod}_{V_{\mathrm{Is}}}$ where V_{Is} is the Ising VOA with the central charge $c = \frac{1}{2}$. It has three simple objects $\mathbb{1}, \psi, \sigma$ (i.e. $\mathbb{1} = V_{\mathrm{Is}}$) and the following fusion rules:
  $$\psi \otimes \psi = \mathbb{1}, \quad \psi \otimes \sigma = \sigma, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$
  2. $\mathfrak{Z}_1(\mathrm{Is}) \simeq \mathrm{Is} \boxtimes \overline{\mathrm{Is}}$. It has 9 simple objects: $\mathbb{1} \boxtimes \mathbb{1}, \mathbb{1} \boxtimes \psi, \mathbb{1} \boxtimes \sigma, \psi \boxtimes \mathbb{1}, \cdots$.
  3. $\mathrm{TC} =$ the MTC of toric code. It has four simple objects $1, e, m, f$ and the following fusion rules:
  $$e \otimes e = m \otimes m = f \otimes f = 1, \quad m \otimes e = f.$$

- Three non-chiral symmetries (i.e. full field algebras Huang-K.:math/0511328):
  1. $P = V_{\mathrm{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathrm{Is}}} = \mathbb{1} \boxtimes \mathbb{1} \in \mathrm{Is} \boxtimes \overline{\mathrm{Is}} = \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_P = |\chi_0|^2$;
  2. $Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_Q = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2$
  3. $R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma \in \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_R = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2$

  We have $P \lneq Q \lneq R$.

- P, Q, R are condensable algebras in $\mathfrak{Z}_1(\mathrm{Is})$ and R is a Lagrangian algebra.
  1. $\phi_P : \mathrm{Mod}_P = (\mathfrak{Z}_1(\mathrm{Is}))_P^0 \xrightarrow{\simeq} \mathfrak{Z}_1(\mathrm{Is})$: condensing P gives $\mathfrak{Z}_1(\mathrm{Is})$;
  2. $\phi_Q : \mathrm{Mod}_Q = (\mathfrak{Z}_1(\mathrm{Is}))_Q^0 \xrightarrow{\simeq} \mathrm{TC}$: condensing Q in $\mathfrak{Z}_1(\mathrm{Is})$ gives toric code TC Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334;
  3. $\phi_R : \mathrm{Mod}_R = (\mathfrak{Z}_1(\mathrm{Is}))_R^0 \xrightarrow{\simeq} \mathrm{Vec}$: condensing R in $\mathfrak{Z}_1(\mathrm{Is})$ gives the trivial phase.

$$P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1}$$
$$Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi$$
$$R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma$$
$$\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$$
$$\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathsf{TC}$$
$$\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$$



- Four anomaly-free 1+1D gapless quantum liquids defined by triples: i.e. its 2+1D bulk topological order is trivial: $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$ and $\mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S}) \simeq \mathrm{Vec}$.
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathsf{Is})$: in this case $V = P \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2))$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  3. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2})$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})$: in this case $V = R = M_{\mathbb{1},\mathbb{1}}$.

  In all 4 cases, the space of non-chiral fields living on each 2-cells (i.e. $M_{\mathbb{1},\mathbb{1}}$) is given by the same modular-invariant closed CFT R.

- Gappable gapless edges of 2+1D toric code: $\mathsf{TC} = \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$, Chen-Jian-K.-You-Zheng:1903.12334, K.-Zheng:1912.01760 (skip unless people ask questions)
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathcal{S})$, where $\mathcal{S} = (\mathfrak{Z}_1(\mathsf{Is}))_Q$ is the fusion category of the right Q-modules in $\mathfrak{Z}_1(\mathsf{Is})$. $\mathcal{S}$ has 6 simple objects $\mathbb{1}, e, m, f, \chi_\pm$, where $\mathbb{1}, e, m, f$ can be identified with 4 anyons in the bulk and $\chi_\pm$ can be identified with two twist defects in the bulk.
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathsf{TC})$ = the canoncial gapless edge;
  3. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Rep}(\mathbb{Z}_2))$, where $\phi : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$. Moreover, we have
$$(R, \phi_R, \mathrm{Rep}(\mathbb{Z}_2)) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Rep}(\mathbb{Z}_2))}_{\text{the smooth gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}).}_{\text{an anomaly-free 2D gapless liquid}}$$
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}_{\mathbb{Z}_2})$:
$$(R, \phi_R, \mathrm{Vec}_{\mathbb{Z}_2}) = \underbrace{(\mathbb{C}, \mathsf{id}, \mathrm{Vec}_{\mathbb{Z}_2})}_{\text{the rough gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}$$
  5. (smooth/rough gapped edge) $\boxtimes$ (any anomaly-free 2D gapless liquid).