Higher Condensation Theory

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Applications of Generalized Symmetries and Topological Defects to Quantum Matter: September 9-13, 2024, Simons Center

References: joint with Zhi-Hao Zhang, Jia-Heng Zhao, Hao Zheng, arXiv:2403.07813 (upcoming 2nd version!)

Anyon condensations in 2+1D

The mathematical theory of anyon condensation has a long history Moore-Seiberg:1988-1989, Bais-Slingerland:2002-2008, Kapustin-Saulina:1008.0654, Levin:1301.7355, Barkeshli-Jian-Qi:1305.7203, ..., Böckenhauer-Evans-Kawahigashi:math/9904109,0002154, Kirillov-Ostrik:math/0101219, Frölich-Fuchs-Runkel-Schweigert:math/0309465, K.:1307.8244, ...

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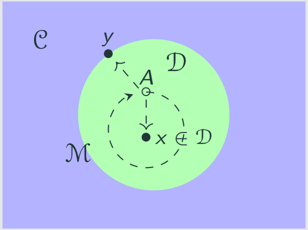
Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be the modular tensor categories (MTC) of anyons in two 2+1D topological orders. An anyon (or boson) condensation from $\mathcal{C}$ to $\mathcal{D}$, which produces a gapped domain wall $\mathcal{M}$, is determined by a condensable E_2-algebra A in $\mathcal{C}$.

- *$\mathcal{D}$ is the category of deconfined particles =the category of local A-modules (or E_2-A-modules) in $\mathcal{C}$; the trivial anyon $\mathbb{1}_{\mathcal{D}} \in \mathcal{D}$ is $A \in \mathcal{C}$;* $\otimes_{\mathcal{D}} = \otimes_A$.
- *$\mathcal{M}$ is the category of (de)-confined particles = the category of right A-modules in $\mathcal{C}$.*
- *bulk-to-wall maps:* $\mathcal{C} \xrightarrow{-\otimes A} \mathcal{M} \hookleftarrow \mathcal{D}$



Example:

1. Consider 2+1D Ising topological order Is[3], we use $\mathcal{I}s$ to denote the category of all topological defects (codimension 1 and higher), and use $\Omega\mathcal{I}s$ to denote that of defects of codimension 2 and higher, i.e., the MTC of anyons.

   $\Omega\mathcal{I}s$ has three simple objects $\mathbb{1}, \psi, \sigma$. Consider the double Ising MTC.

$$\mathfrak{Z}_1(\Omega\mathcal{I}s) = \Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}} = \{\mathbb{1}\boxtimes\mathbb{1}, \mathbb{1}\boxtimes\psi, \psi\boxtimes\mathbb{1}, \psi\boxtimes\psi, \sigma\boxtimes\psi, \sigma\boxtimes\sigma, \cdots\}$$

2. Toric code MTC: $\Omega\mathcal{TC} = \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$ consisting of four simple objects $1, e, m, f$.

Now we consider the condensable E_2-algebra in $\Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}}$:

$$A := \mathbb{1}\boxtimes\mathbb{1} \oplus \psi\boxtimes\psi \quad \in \Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}}. \qquad (1)$$

By condensing A in $\mathcal{C}$, we obtain toric code from double Ising. Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334.

$$\Omega\mathcal{TC} \simeq (\Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}})_A^{loc} = \mathrm{Mod}_A^{\mathrm{E}_2}(\Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}}).$$

Contrary to the physical intuitions that a phase transition between two phases are reversible, above mathematical description of an anyon condensation is not reversible because we have Davydov-Müger-Nikshych-Ostrik:1009.2117.

$$\dim \mathcal{D} = \frac{\dim \mathcal{C}}{(\dim A)^2},$$

where $\dim A > 1$ for any non-trivial condensation.

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It turns out that this phenomenon is a reflection of the fact that the 1-category of anyons (i.e., defects of codimension 2) does not include all topological defects in a 2+1D topological order Kitaev-K.:1104.5047. In this talk, I will show that, by including all topological defects of codimension 1 and higher, we obtain a 2-category and a rather complete defect condensation theory, which is ready to be generalized to higher dimensions.

Category of topological defects

The mathematical theory of topological defects in an $n+1$D (potentially anomalous) topological order was developed in a series of works based on three guiding principles.

1. Remote Detectable Principle, Levin:1301.7355, K.-Wen:1405.5858
2. Boundary-bulk relation, Kitaev-K.:1104.5047, K.-Wen-Zheng:1502.01690,1702.00673
3. Condensation Completion Principle. Carqueville-Runkel:1210.6363, Douglas-Reutter:1812.11933, Gaiotto, Johnson-Freyd:1905.09566, Johnson-Freyd:2003.06663, K.-Lan-Wen-Zhang-Zheng:2003.08898

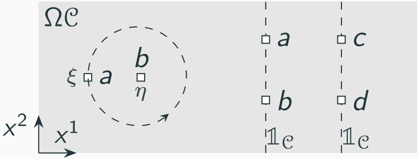
Preceded by some earlier attempts K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, this theory was established by Johnson-Freyd in 2020 Johnson-Freyd:2003.06663 (see further developments in K.-Zheng:2011.02859,2107.03858).

We summarize the main results of this theory for an (potentially anomalous) n+1D topological order C^{n+1}.

1. The category of all topological defects (of codimension 1 and higher) form a fusion n-category $\mathcal{C}$ (an E_1-fusion n-category). Johnson-Freyd:2003.06663
   1.1 0-morphisms (i.e., objects) in $\mathcal{C}$ are 1-codimensional defects;
   1.2 1-morphisms in $\mathcal{C}$ are 2-codimensional defects;
   1.3 k-morphisms are $(k+1)$-codimensional defects;
   1.4 n-morphisms are $(n+1)$-codimensional defects (i.e., instantons or 0D defects).
2. $\mathbb{1} \in \mathcal{C}$ labels the trivial 1-codimensional defect.
3. $\Omega\mathcal{C} := \mathrm{hom}(\mathbb{1}, \mathbb{1})$ is the category of defects of codimension 2 and higher. It is braided fusion $(n-1)$-category or E_2-fusion $(n-1)$-category.
4. $\Omega^{k-1}\mathcal{C}$ is the category of defects of codimension k and higher. It is an E_k-fusion $(n-k+1)$-category. K.-Zheng:2011.02859

I will explain the meaning of 'E_1', 'E_2' and 'E_k'.

1. $\mathcal{C}$ is E_1-fusion: two 1-codimensional defects $x, y \in \mathcal{C}$ can be fusion in one direction $x \otimes^1 y$.
2. $\Omega\mathcal{C}$ is E_2-fusion: two 2-codimensional defects $a, b \in \Omega\mathcal{C}$ can be fused in two orthogonal directions:



$a \otimes^1 c$
$b \otimes^2 a$
$\Leftrightarrow$ braiding structure

3. $\Omega^{k-1}\mathcal{C}$ is E_k-fusion: two k-codimensional defects $p, q \in \Omega^{k-1}\mathcal{C}$ can be fused in k orthogonal directions: $p \otimes^1 q, p \otimes^2 q, \cdots, p \otimes^k q$.

Convention of notations:

- Topological orders: $\mathsf{A}^{n+1}, \mathsf{B}^{n+1}, \mathsf{C}^{n+1}, \cdots$;
- Category of all defects (codimension 1 or higher): $\mathcal{A}, \mathcal{B}, \mathcal{C}, \cdots$;
- Category of defects of codimension 2 or higher: $\Omega\mathcal{A}, \Omega\mathcal{B}, \Omega\mathcal{C}, \cdots$;
- Category of defects of codimension k or higher: $\Omega^{k-1}\mathcal{A}, \Omega^{k-1}\mathcal{B}, \Omega^{k-1}\mathcal{C}, \cdots$.

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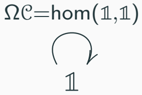
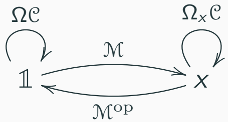
If C^{n+1} is anomaly-free, then $\mathcal{C} = \Sigma\Omega\mathcal{C} = \mathrm{Kar}(\mathrm{B}\Omega\mathcal{C})$ ($n=2$ Carqueville-Runkel:1210.6363, Douglas-Reutter:1812.11933; $n \geq 2$ Gaiotto, Johnson-Freyd:1905.09566).

$\mathrm{B}\Omega\mathcal{C} \simeq$ 

$\Sigma\Omega\mathcal{C} = \mathrm{Kar}(\mathrm{B}\Omega\mathcal{C}) =$ 

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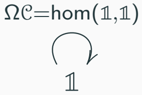
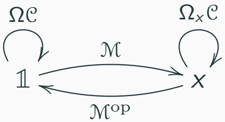
$\mathrm{B}\Omega\mathcal{C} \simeq$ 

$\Sigma\Omega\mathcal{C} = \mathrm{Kar}(\mathrm{B}\Omega\mathcal{C}) =$ 

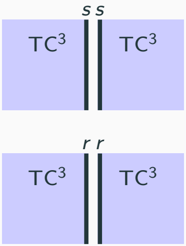
Physically, $\mathcal{C} = \Sigma\Omega\mathcal{C}$ means that all 1-codimensional defects, which cannot be braided, can only be the condensation descendants of 2-codimensional defects, which can be braided and detectable by double braidings. K.-Wen:1405.5858.

2+1D $\mathbb{Z}_2$ topological order TC^3: We denote the fusion 2-category of topological defects in TC^3 by $\mathcal{TC}$. In this case, $\Omega\mathcal{TC}$ has four simple objects (or anyons) $1, e, m, f$:

$$e \otimes e \simeq m \otimes m \simeq f \otimes f \simeq 1, \qquad f \simeq e \otimes m \simeq m \otimes e.$$

Mathematically, $\Omega\mathcal{TC}$ can be identified with the Drinfeld center $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$ of $\mathrm{Rep}(\mathbb{Z}_2)$. The fusion 2-category $\mathcal{TC}$ consists of six simple objects (i.e. 1-codimensional topological defects) $\mathbb{1}, \theta, \mathrm{ss}, \mathrm{sr}, \mathrm{rs}, \mathrm{rr}$. K.-Zhang:2205.05565 θ is the invertible domain wall realizing the e-m duality.

$\otimes$	$\mathbb{1}$	θ	ss	sr	rs	rr
$\mathbb{1}$	$\mathbb{1}$	θ	ss	sr	rs	rr
θ	θ	$\mathbb{1}$	rs	rr	ss	sr
ss	ss	sr	2ss	2sr	ss	sr
sr	sr	ss	ss	sr	2ss	2sr
rs	rs	rr	2rs	2rr	rs	rr
rr	rr	rs	rs	rr	2rs	2rr



2+1D Ising topological order Is^3: We denote the fusion 2-category of topological defects in Is^3 by $\mathcal{I}s$. In this case, $\Omega\mathcal{I}s = \{1, \psi, \sigma\}$ is the MTC of anyons.

It turns out that the only simple 1-codimensional topological defect in Is^3 is the trivial defect $\mathbb{1}$. Fuchs-Runkel-Schweigert:hep-th/0204148.

$$\mathcal{I}s \simeq \Sigma\Omega\mathcal{I}s \simeq \mathrm{B}\Omega\mathcal{I}s \simeq \overset{\Omega\mathcal{I}s}{\underset{\mathbb{1}}{\circlearrowright}}$$

Theorem Gaiotto, Johnson-Freyd:1905.09566: We denote the category of defects of the trivial topological order $\mathbf{1}^{n+1}$ by $n\mathrm{Vec}$, which consists of only trivial k-dimensional defects $\mathbf{1}^k$ for $1 \leq k \leq n$ and their condensation descendants (i.e., defects that can be obtained from $\mathbf{1}^k$ via condensation).

(1) $\mathbf{1}^{0+1}$: $0\mathrm{Vec} = \mathbb{C}$ (i.e., trivial fusion 0-category);

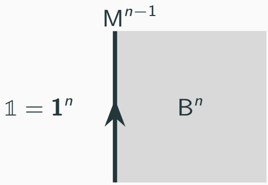
(2) $\mathbf{1}^{1+1}$: $1\mathrm{Vec} := \Sigma\mathbb{C} = \mathrm{Kar}(\mathrm{B}\mathbb{C}) = \{\mathbb{C}^{\oplus k}\}$ (i.e., trivial fusion 1-category);

$$\mathbb{1} = \mathbf{1}^1,$$

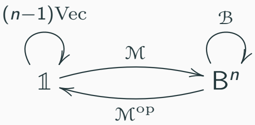
(3) $\mathbf{1}^{2+1}$: $2\mathrm{Vec} = \Sigma\mathrm{Vec} = \mathrm{Kar}(\mathrm{B}\mathrm{Vec}) = \{\mathrm{Vec}^{\oplus k}\}$ (i.e., trivial fusion 2-category);

$$\mathbb{1} = \mathbf{1}^2,$$

(4) $\mathbf{1}^{n+1}$: $n\mathrm{Vec} = \Sigma(n-1)\mathrm{Vec} = \Sigma^n\mathbb{C} \neq \{(n-1)\mathrm{Vec}^{\oplus k}\}$ for $n \geq 3$. For example, $3\mathrm{Vec} = \{2\mathrm{Vec}^{\oplus k}, \Sigma\mathcal{A} \,|\, \mathcal{A}$ is a multi-fusion 1-category$\} = \{$separable 2-categories$\}$.

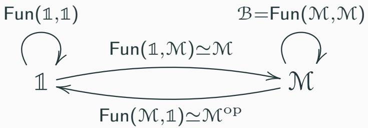


A non-chiral topological order B^n (i.e., admits a gapped boundary) is a condensation descendant of $\mathbb{1} = \mathbf{1}^n$



$$n\mathrm{Vec} \xrightarrow{\simeq} \mathcal{S}ep_{n-1}$$
$$\mathbb{1} \mapsto (n-1)\mathrm{Vec}$$
$$\mathrm{B}^n \mapsto \mathcal{M}$$

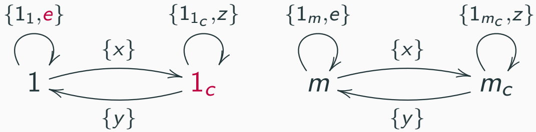
K.-Zheng:2011.02859



Remark: The physical meaning of the $(n-1)$-category $\mathcal{M}$ is the category of gapped boundary conditions of B^n.

3+1D toric code TC^4: According to our convention of notations, we use $\mathcal{TC}^4$ to denote the category of all defects. It is a fusion 3-category which contains infinitely many simple objects (i.e., 1-codimensional defects).

1. It is relatively easier to describe the braided fusion 2-category $\Omega\mathcal{TC}^4$ of the defects of codimension 2 and higher. There are four simple 2-codimensional defects: $1, 1_c, m, m_c$,



   where 1_c is the condensation descendant of 1 (by condensing the e-particles along a line K.-Wen:1405.5858) and is sometimes called a Cheshire string. Else-Nayak:1702.02148

2. fusion rules: $m \otimes m = 1$, $1_c \otimes 1_c = 1_c \oplus 1_c$ and $m \otimes 1_c = m_c$.
   K.-Tian-Zhou:1905.04644, K.-Tian-Zhang:2009.06564

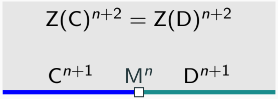
Condensations of topological defects

Theorem (K.-Zhang-Zheng-Zhao:2403.07813): Condensing a k-codimensional topological defect A in an n+1D (potentially anomalous) topological order C^{n+1} amounts to a k-step process.

(1) The k-codimensional defect $A \in \Omega^{k-1}\mathcal{C}$ is condensable if it is equipped with the structure of a condensable E_k-algebra, i.e. an algebra equipped with compatible multiplications in k independent directions.
We first condense A along one of the transversal directions x^k, thus obtaining a $(k-1)$-codimensional defect $\Sigma A := \mathrm{RMod}_A(\Omega^{k-1}\mathcal{C}) \in \Sigma\Omega\mathcal{C}$.

(2) It turns out that ΣA is naturally equipped with the structure of a condensable E_{k-1}-algebra, thus it can be further condensed along one of the remaining transversal direction x^{k-1}, thus obtaining a $(k-2)$-codimensional defect $\Sigma^2 A := \mathrm{RMod}_{\Sigma A}(\Omega^{k-2}\mathcal{C})$.

(3) In the k-th step, condensing the 1-codimensional defect $\Sigma^{k-1}A$ along the only transversal direction defines a phase transition to a new n+1D topological order D^{n+1}, which is Morita equivalent to C^{n+1}, and a gapped domain wall M^n.



$$\mathcal{D} \simeq \mathrm{Mod}^{E_1}_{\Sigma^{k-1}A}(\mathcal{C}) = \mathrm{BMod}_{\Sigma^{k-1}A|\Sigma^{k-1}A}(\mathcal{C})$$
$$(\mathcal{M}, m) = (\Sigma^k A := \mathrm{RMod}_{\Sigma^{k-1}A}(\mathcal{C}), \Sigma^{k-1}A).$$

(4) A k-codimensional deconfined topological defects in D^{n+1} form the category $\Omega^{k-1}\mathcal{D}$, which can be computed directly as the category of E_k-A-modules.

$$\Omega^{k-1}\mathcal{D} = \Omega^{k-1}\,\mathrm{Mod}^{\mathrm{E}_1}_{\Sigma^{k-1}A}(\mathcal{C}) \simeq \mathrm{Mod}^{\mathrm{E}_k}_{A}(\Omega^{k-1}\mathcal{C}),$$

A k-codimensional topological defect is deconfined iff it is equipped with a 'k-dimensional A-action', which defines the mathematical notion called an E_k-module over A or an E_k-A-module.

(5) Similarly, the confined k-codimensional defects (confined to the wall M^n) can also be computed directly.

$$\Omega^{k-1}_m\mathcal{M} = \Omega^{k-1}\Sigma^k A = \Omega^{k-1}\,\mathrm{RMod}_{\Sigma^{k-1}A}(\mathcal{C}) \simeq \mathrm{RMod}_A(\Omega^{k-1}\mathcal{C}).$$

(6) When C^{n+1} is anomaly-free (i.e., $\mathcal{C} = \Sigma\Omega\mathcal{C}$), the same phase transition, as a k-step process, can be alternatively defined by replacing the last two steps by a single step of condensing the E_2-algebra $\Sigma^{k-2}A$ in the remaining two transversal directions directly.

The condensed phase D^{n+1} is determined by the category of E_2-modules over $\Sigma^{k-2}A$.

$$\Omega\mathcal{D} \simeq \mathsf{Mod}^{\mathrm{E}_2}_{\Sigma^{k-2}A}(\Omega\mathcal{C}), \qquad \mathcal{D} \simeq \Sigma\Omega\mathcal{D}, \qquad \Omega_m\mathcal{M} = \mathsf{RMod}_{\Sigma^{k-2}A}(\Omega\mathcal{C}).$$

When $n = 2$, this modified last step is precisely a usual anyon condensation in 2+1D.

(7) The condensable E_k-algebra A is called Lagrangian if $\mathsf{D}^{n+1} = \mathbf{1}^{n+1}$.

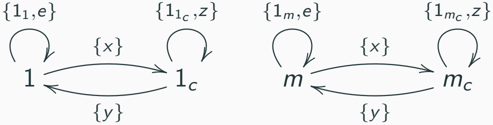
Example: Consider 2+1D $\mathbb{Z}_2$ topological order TC^3.

1. $A_e = 1 \oplus e$ is a Lagrangian E_2-algebra in $\Omega\mathcal{TC}$. Condensing it along a line produces a string $\Sigma A_e = \mathrm{rr}$, which can be further condensed to create the rough boundary of TC^3.

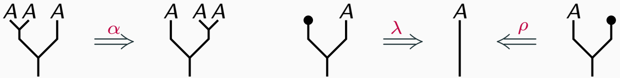
2. $A_m = 1 \oplus m$ is a Lagrangian E_2-algebra in $\Omega\mathcal{TC}$. Condensing it along a line produces a string $\Sigma A_m = \mathrm{ss}$, which can be further condensed to create the smooth boundary of TC^3.

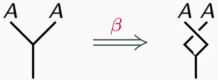
Consider 3+1D toric code TC^4

the braided fusion 2-category $\Omega\mathcal{TC}^4$:



A commutative algebra (or an E_2-algebra) in $\Omega\mathcal{TC}^4$ is an object A equipped with two 1-morphisms: $A \otimes A \xrightarrow{\mu} A$ and $1 \xrightarrow{\eta} A$, and three 2-morphisms: left/right unitors λ, ρ, associator α and commutator β.





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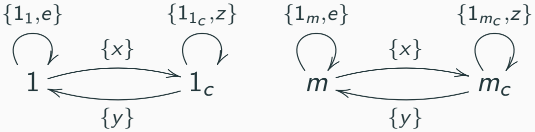
the braided fusion 2-category $\Omega\mathcal{TC}^4$:



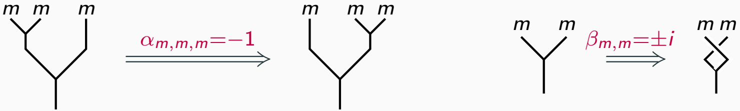
There are three Lagrangian E_2-algebras (among infinitely many) in $\Omega\mathcal{TC}^4$.

(1) $A_e = 1_c$: $\quad 1 \xrightarrow{\eta = x} 1_c, \quad 1_c \otimes 1_c = 1_c \oplus 1_c \xrightarrow{\mu = 1_{1_c} \oplus 0} 1_c.$

(2) $A_m := 1 \oplus m$

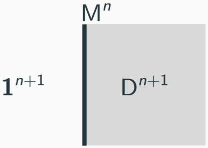
$$\begin{array}{c} A_m \otimes A_m \\ \downarrow \mu \\ A_m \end{array} \; := \; \begin{array}{c} 1 \otimes 1 \\ \downarrow 1_1 \\ 1 \end{array} \quad \begin{array}{c} 1 \otimes m \\ \downarrow 1_m \\ m \end{array} \quad \begin{array}{c} m \otimes 1 \\ \downarrow 1_m \\ m \end{array} \quad \begin{array}{c} m \otimes m \\ \downarrow 1_1 \\ 1 \end{array}$$

(3) $A_m^{tw} = (1 \oplus m)^{tw}$:



Examples: $\mathrm{C}^{n+1} = \mathbf{1}^{n+1}$. In this case, the category of all defects $\mathcal{C} = n\mathrm{Vec}$ (an E_1-fusion n-category).

A condensable E_1-algebra $\mathcal{A}$ in $n\mathrm{Vec}$ is precisely a multi-fusion $(n-1)$-category. By condensing $\mathcal{A}$ in $n\mathrm{Vec}$, we obtain the condensed phase D^{n+1} and a gapped domain wall M^n.



D^{n+1} is a non-chiral topological order (i.e., admitting a gapped boundary) and a condensation descendant of $\mathbf{1}^{n+1}$

$$\mathcal{D} \simeq \mathrm{BMod}_{\mathcal{A}|\mathcal{A}}(n\mathrm{Vec}), \qquad \mathrm{M}^n = (\mathcal{M}, m) = (\mathrm{RMod}_{\mathcal{A}}(n\mathrm{Vec}), \mathcal{A})$$

$$\Omega\mathcal{D} = \mathrm{Fun}_{\mathcal{A}|\mathcal{A}}(\mathcal{A}, \mathcal{A}) = \mathfrak{Z}_1(\mathcal{A}), \qquad \Omega_m\mathcal{M} = \mathcal{A}$$

If $\mathfrak{Z}_1(\mathcal{A}) = (n-1)\mathrm{Vec}$, then $\mathcal{D} = \Sigma\Omega\mathcal{D} = \Sigma(n-1)\mathrm{Vec} = n\mathrm{Vec}$ (i.e., $\mathrm{D}^{n+1} = \mathbf{1}^{n+1}$). It means that a multi-fusion $(n-1)$-category $\mathcal{A}$ is Lagrangian iff $\mathfrak{Z}_1(\mathcal{A}) = (n-1)\mathrm{Vec}$.

4 Since $\mathsf{C}^{n+1} = \mathbf{1}^{n+1}$ is anomaly-free, we can also condense a defect of codimension 2 directly. The category of 2-codimensional defects in $\mathbf{1}^{n+1}$ is $(n-1)\mathrm{Vec}$. A condensable E_2-algebra in $(n-1)\mathrm{Vec}$ is precisely a braided fusion $(n-2)$-category $\mathcal{B}$. By condensing $\mathcal{B}$ directly along remaining two transversal directions, we obtain D^{n+1} and a gapped domain wall M^n.

$\Omega_m\mathcal{M}$

$\Omega\mathcal{C} = (n-1)\mathrm{Vec}$

D^{n+1} is a non-chiral topological order (i.e., admitting a gapped boundary) and a condensation descendant of $\mathbf{1}^{n+1}$

$$\Omega\mathcal{D} = \mathrm{Mod}_{\mathcal{B}}^{\mathrm{E}_2}((n-1)\mathrm{Vec}), \qquad \mathcal{D} = \Sigma\Omega\mathcal{D} \simeq \Sigma\,\mathrm{Mod}_{\mathcal{B}}^{\mathrm{E}_2}((n-1)\mathrm{Vec}) \simeq \mathrm{Mod}_{\Sigma\mathcal{B}}^{\mathrm{E}_1}(n\mathrm{Vec}),$$

$$\Omega_m\mathcal{M} = \mathrm{RMod}_{\mathcal{B}}((n-1)\mathrm{Vec}) = \Sigma\mathcal{B}, \qquad \mathcal{M} = \Sigma\Omega_m\mathcal{M}.$$

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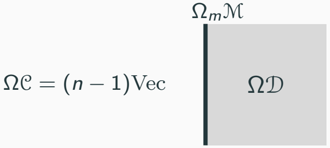

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$$\Omega_m\mathcal{M} = \mathrm{RMod}_{\mathcal{B}}((n-1)\mathrm{Vec}) = \Sigma\mathcal{B}, \qquad \mathcal{M} = \Sigma\Omega_m\mathcal{M}.$$

When $n = 3$, every fusion 2-category is Morita equivalent to $\Sigma\mathcal{A}$ for a braided fusion 1-category $\mathcal{A}$. Décoppet:2208.08722 It means that all non-chiral 3+1D topological orders can be obtained from $\mathbf{1}^4$ by condensing defects of codimension 2 (i.e., strings).

When $\mathcal{C}^{n+1} = \mathcal{D}^{n+1} = \mathbf{1}^{3+1}$, the category of strings is $\Omega\mathcal{C} = 2\mathrm{Vec}$. The gapped domain wall M^3 is precisely a 2+1D anomaly-free topological order, which can described by a pair $(\mathcal{B}, c)$ for a modular tensor category $\mathcal{B}$.

By rolling up the 2+1D anomaly-free topological order M^3, we obtain a string-like defect in $\mathbf{1}^4$, which is precisely the object $\mathcal{B} \in \Omega\mathcal{C} = 2\mathrm{Vec}$. By condensing this string, we obtain $\Omega\mathcal{D} = \mathrm{Mod}_{\mathcal{B}}^{E_2}(2\mathrm{Vec}) \simeq \Omega\,\mathrm{Mod}_{\Sigma\mathcal{B}}^{E_1}(3\mathrm{Vec}) \simeq \Omega 3\mathrm{Vec} = 2\mathrm{Vec}$. Therefore, $\mathcal{D}^4 = \mathbf{1}^4$ and $\mathcal{B}$ is a Lagrangian E_2-algebra in $2\mathrm{Vec}$.

(3). C^{n+1} is anomaly-free, i.e. $\mathfrak{Z}_1(\mathcal{C}) = n\mathrm{Vec}$. In this case, we have

$$\mathcal{C} = \Sigma\Omega\mathcal{C} = \mathrm{RMod}_{\Omega\mathcal{C}}(n\mathrm{Vec})$$

provides a concrete coordinate system to the non-degenerate fusion n-category $\mathcal{C}$.

Theorem (Brochier-Jordan-Synder:1804.07538, K.-Zhang-Zhao-Zheng:2403.07813)

An indecomposable condensable E_1-algebra in $\mathcal{C}$ are precisely an indcomposable multi-fusion $(n-1)$-category $\mathcal{A}$ equipped with a central functor $L : \Omega\mathcal{C} \to \mathcal{A}$ (i.e., a braided monoidal functor $\phi : \Omega\mathcal{C} \to \mathfrak{Z}_1(\mathcal{A})$). When $\phi : \Omega\mathcal{C} \to \mathfrak{Z}_1(\mathcal{A})$ is a braided equivalence, the condensable E_1-algebra $\mathcal{A}$ is Lagrangian in the sense that $\mathrm{D}^{n+1} = \mathbf{1}^{n+1}$.

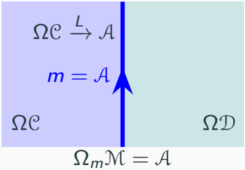
Condensing $\mathcal{A}$ in $\mathcal{C}$ produces a condensed phase D^{n+1} and a gapped domain wall M^n:

$$\mathcal{D} \simeq \mathrm{Mod}_{\mathcal{A}}^{\mathrm{E}_1}(\mathcal{C}), \qquad \mathcal{M} \simeq \mathrm{RMod}_{\mathcal{A}}(\mathcal{C}) \simeq \mathrm{RMod}_{\mathcal{A}}(n\mathrm{Vec}), \qquad m = \mathcal{A},$$

Question: Does the abstract data of $L : \Omega\mathcal{C} \to \mathcal{A}$, which defines an abstract algebra, has a direct physical meaning?

Physical Meaning of this Theorem:

$$\mathcal{M} \simeq \mathrm{RMod}_{\mathcal{A}}(\mathcal{C}) \simeq \mathrm{RMod}_{\mathcal{A}}(n\mathrm{Vec}),$$
$$m = \mathcal{A},$$
$$\Omega_m \mathcal{M} = \mathrm{RMod}_{\mathcal{A}}(\mathcal{A}, \mathcal{A}) \simeq \mathcal{A}.$$



Take home message: The multi-fusion higher category $\Omega_m\mathcal{M}$ (of all defects on the domain wall M^n) is a condensable E_1-algebra in $\mathcal{C}$.

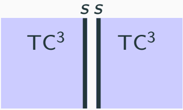
Example: $C^3 = TC^3$: The topological defects in TC^3 form the fusion 2-category

$$\mathcal{TC} \simeq \Sigma(\Omega\mathcal{TC}) = \mathsf{RMod}_{\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))}(2\mathrm{Vec}).$$

A condensable E_1-algebra in $\mathcal{TC}$ are precisely a mulit-fusion categories $\mathcal{A}$ equipped with a central functor $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \to \mathcal{A}$.

1. $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \xrightarrow{\mathrm{id}} \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$ defines a condensable E_1-algebra $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))$. It is just the trivial condensable E_1-algebra, i.e., tensor unit $\mathbb{1}$ of $\mathcal{TC}$.
2. $\mathsf{f} : \mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \to \mathrm{Rep}(\mathbb{Z}_2)$ defines a condensable E_1-algebra $\mathrm{Rep}(\mathbb{Z}_2)^{\mathrm{op}} = \mathrm{Rep}(\mathbb{Z}_2)$ in $\mathcal{TC}$, which is precisely the 1-codimensional defect ss in $\mathcal{TC}$.

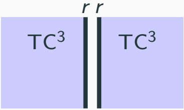
$$\mathsf{Mod}^{\mathrm{E}_1}_{\mathrm{Rep}(\mathbb{Z}_2)}(\mathcal{TC}) \simeq 2\mathrm{Vec},$$



Since the condensed phase is trivial, ss is a Lagrangian condensable E_1-algebra in $\mathcal{TC}$.

3 $\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2)) \xrightarrow{\simeq} \mathfrak{Z}_1(\mathrm{Vec}_{\mathbb{Z}_2}) \to \mathrm{Vec}_{\mathbb{Z}_2}$ defines a condensable E_1-algebra $\mathrm{Vec}_{\mathbb{Z}_2}^{\mathrm{op}} = \mathrm{Vec}_{\mathbb{Z}_2}$, which is precisely the 1-codimensional defect rr in $\mathcal{TC}$. We have

$$\mathrm{Mod}^{\mathrm{E}_1}_{\mathrm{Vec}_{\mathbb{Z}_2}}(\Sigma\mathfrak{Z}_1(\mathrm{Rep}(\mathbb{Z}_2))) \simeq 2\mathrm{Vec},$$



Since the condensed phase is trivial, rr is a Lagrangian condensable E_1-algebra in $\mathcal{TC}$.

4 Consider double Ising 2+1D topological order ($\Omega\mathcal{I}s$ is the Ising MTC).

$$A := \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \Omega\mathcal{I}s \boxtimes \Omega\mathcal{I}s^{\mathrm{op}} \simeq \mathfrak{Z}_1(\Omega\mathcal{I}s) \qquad (2)$$

has a condensable E_2-algebra in $\mathfrak{Z}_1(\Omega\mathcal{I}s)$. Then the central functor $\{1, e, m, f\} = \Omega\mathcal{TC} = \mathrm{Mod}^{\mathrm{E}_2}_A(\mathfrak{Z}_1(\Omega\mathcal{I}s)) \hookrightarrow \mathrm{RMod}_A(\mathfrak{Z}_1(\Omega\mathcal{I}s))^{\mathrm{op}} = \{1, e, m, f, \chi_\pm\} = \mathcal{K}$ Chen-Jian-K.-You-Zheng:1903.12334 defines a condensable E_1-algebra in $\mathcal{TC}$. Condensing it in $\mathcal{TC}$ produces the 2+1D double Ising topological order as the condensed phase.

$$\mathrm{Mod}^{\mathrm{E}_1}_{\mathcal{K}}(\mathcal{TC}) \simeq \mathcal{I}s \boxtimes \mathcal{I}s^{\mathrm{op}}.$$

(4). C^{n+1} has a gapped boundary B^n, i.e., $\mathsf{C}^{n+1} = \mathsf{Z}(\mathsf{B})^{n+1}$: In this case, we have a new coordinate system for $\mathcal{C}$ (obtained by condensing $\mathcal{B}^{\mathrm{op}}$ in $n\mathrm{Vec}$):

$$\mathcal{C} = \mathsf{BMod}_{\mathcal{B}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}}(n\mathrm{Vec}) \simeq \mathsf{BMod}_{\mathcal{B}|\mathcal{B}}(n\mathrm{Vec})^{\mathrm{op}}$$

1. An indecomposable condensable E_1-algebras in $\mathcal{C}$ is precisely an indecomposable multi-fusion n-category $\mathcal{A}^{\mathrm{op}}$ equipped with a monoidal functor $\mathcal{B} \to \mathcal{A}$.
2. By condensing $\mathcal{A}^{\mathrm{op}}$ in $\mathcal{C}$, we obtain the condensed topological order $\mathsf{D}^{n+1} = \mathsf{Z}(\mathsf{A})^{n+1}$ with

$$\mathcal{D} \simeq \mathsf{Mod}^{\mathrm{E}_1}_{\mathcal{A}^{\mathrm{op}}}(\mathcal{C}) \simeq \mathsf{BMod}_{\mathcal{A}^{\mathrm{op}}|\mathcal{A}^{\mathrm{op}}}(\mathsf{BMod}_{\mathcal{B}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}}(n\mathrm{Vec})) \simeq \mathsf{Mod}^{\mathrm{E}_1}_{\mathcal{A}^{\mathrm{op}}}(n\mathrm{Vec}) \simeq \Sigma\mathfrak{Z}_1(\mathcal{A})$$

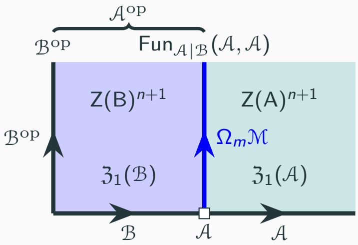
and a gapped domain wall M^n

$$\mathcal{M} \simeq \mathsf{RMod}_{\mathcal{A}^{\mathrm{op}}}(\mathsf{BMod}_{\mathcal{B}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}}(n\mathrm{Vec})) \simeq \mathsf{BMod}_{\mathcal{A}|\mathcal{B}}(n\mathrm{Vec}), \quad m = \mathcal{A}, \quad \Omega_m\mathcal{M} = \mathsf{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{A},\mathcal{A}).$$

For $\mathcal{X} \in \mathsf{LMod}_{\mathcal{B}}(n\mathrm{Vec})$, the canonical monoidal functor $\mathcal{B} \to \mathsf{Fun}(\mathcal{X},\mathcal{X})$ defines a Lagrangian E_1-algebra $\mathsf{Fun}(\mathcal{X},\mathcal{X})^{\mathrm{op}}$ in $\mathcal{C} = \mathsf{BMod}_{\mathcal{B}|\mathcal{B}}(n\mathrm{Vec})^{\mathrm{op}}$.

When $\mathsf{C}^{n+1} = \mathsf{Z}(\mathsf{B})^{n+1}$, there are two coordinate systems for $\mathcal{C}$.

$$\mathcal{C} = \Sigma\Omega\mathcal{C} = \mathrm{RMod}_{\Omega\mathcal{C}}(n\mathrm{Vec}), \qquad \mathcal{C} = \mathrm{BMod}_{\mathcal{B}^{\mathrm{op}}|\mathcal{B}^{\mathrm{op}}}(n\mathrm{Vec}).$$

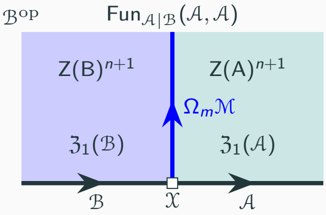


The same algebra in two coordinate systems :

(1) the central functor $\mathfrak{Z}_1(\mathcal{B}) \to \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{A},\mathcal{A})$ defines an algebra $\mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{A},\mathcal{A})$ in $\mathrm{RMod}_{\Omega\mathcal{C}}(n\mathrm{Vec})$;

(2) the monoidal functor $\mathcal{B} \to \mathcal{A}$ defines an algebra $\mathcal{A}^{\mathrm{op}}$ in $\mathrm{BMod}_{\mathcal{B}|\mathcal{B}}(n\mathrm{Vec})^{\mathrm{op}}$.

K.-Zheng:1307.5956,2107.03858



Center functor :
$\mathcal{B} \mapsto \mathfrak{Z}_1(\mathcal{B})$;
$\mathcal{X} \mapsto \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}(\mathcal{A},\mathcal{A}) = \Omega_m\mathcal{M}$;
is fully faithful when $n = 2$.
is faithful when $n = 3$.
K.-Zheng:1307.5956,2107.03858

This general construction of condensable E_1-algebras in $C^{n+1} = Z(B)^{n+1}$ leads to a classification of condensable E_1-algebras in finite gauge theories in 3D, 4D and many constructions for higher dimensions.

When $\mathcal{B} = 2\mathrm{Rep}(G)$ and $\mathcal{A} = 2\mathrm{Vec}$, this map (for a MTC $\mathcal{E}$ K.-Zheng:1705.01087):

$$ {}_{2\mathrm{Rep}(G)}\mathcal{X} \mapsto \mathrm{Fun}_{2\mathrm{Rep}(G)}(\mathcal{X},\mathcal{X})^{\mathrm{op}} \quad ({}_{2\mathrm{Rep}(G)\boxtimes\Sigma\mathcal{E}}\mathcal{X} \mapsto \mathrm{Fun}_{2\mathrm{Rep}(G)\boxtimes\Sigma\mathcal{E}}(\mathcal{X},\mathcal{X})^{\mathrm{op}}) $$

provides a one-to-one correspondence between $2\mathrm{Rep}(G)$-modules and gapped boundaries (or Lagrangian algebras) within the same Morita class as $2\mathrm{Rep}(G)$ (resp. $2\mathrm{Rep}(G) \boxtimes \Sigma\mathcal{E}$).

Some examples in 3+1D from Décoppet:2205.06453, Décoppet-Xu:2307.02843:

1. Condensable E_3-algebras in $2\mathrm{Vec}$ are symmetric multi-fusion 1-categories, which are automatically E_∞-monoidal.
2. Condensable E_2-algebras in $\mathfrak{Z}_1(2\mathrm{Vec}_G)$ for a finite group G are exactly G-crossed braided multi-fusion 1-categories.
3. Let $\mathcal{B}$ be an E_2-fusion 1-category. A braided multi-fusion 1-category $\mathcal{A}$ equipped with a braided functor $\mathcal{B} \to \mathcal{A}$ is a condensable E_2-algebra in the E_2-fusion 2-category $\mathfrak{Z}_1(\mathrm{RMod}_{\mathcal{B}}(2\mathrm{Vec}))$.

A higher dimensional example: Consider the n+2D G-gauge theory GT_G^{n+2}. The category of 2-codimensional defects in GT_G^{n+2} was conjectured in K.-Tian-Zhou:1905.04644 to be:

$$\Omega\mathcal{GT}_G^{n+2} = \mathfrak{Z}_1(n\mathrm{Rep}(G)) \simeq \mathfrak{Z}_1(n\mathrm{Vec}_G) \simeq \oplus_{[h]\in\mathrm{Cl}}\ n\mathrm{Rep}(C_G(h))$$

The fusion $(n+1)$-category $\mathcal{GT}_G^{n+2}$ has two coordinate systems:

$$\mathcal{GT}_G^{n+2} = \mathrm{RMod}_{\mathfrak{Z}_1(n\mathrm{Rep}(G))}((n+1)\mathrm{Vec}), \qquad \mathcal{GT}_G^{n+2} = \mathrm{BMod}_{n\mathrm{Rep}(G)|n\mathrm{Rep}(G)}((n+1)\mathrm{Vec})^{\mathrm{op}},$$

We have $\Omega\mathcal{GT}_G^{n+2} = \mathfrak{Z}_1(n\mathrm{Rep}(G))$ and

$$\Omega^k\mathcal{GT}_G^{n+2} = (n-k+1)\mathrm{Rep}(G) \quad \text{for } k \geq 2$$

When $k=n$, $\Omega^n\mathcal{GT}_G^{n+2} = \mathrm{Rep}(G)$ is the 1-category of particles.

Let $H < G$ be a subgroup of G. The composite particle

$$A = \mathrm{Fun}(G/H) \quad (i.e., \mathbb{C}\text{-valued functions on } G/H)$$

is an E_{n+1}-algebra in $\Omega^n \mathcal{GT}_G^{n+2} = \mathrm{Rep}(G)$. By condensing the A-particles, we mean the following procedures.

(1) We first condensing the A-particle along a line, we obtain a string

$$\Sigma A = \mathrm{RMod}_A(\mathrm{Rep}(G)) \simeq \mathrm{Rep}(H) \in 2\mathrm{Rep}(G) = \Omega^{n-1}\mathcal{GT}_G^{n+2}.$$

(2) We further condense the ΣA-string along one of the remaining transversal directions, we obtain a membrane:

$$\Sigma^2 A = \mathrm{RMod}_{\Sigma A}(2\mathrm{Rep}(G)) \simeq \mathrm{RMod}_{\Sigma A}(2\mathrm{Vec}) \simeq 2\mathrm{Rep}(H) \in 3Rep(G) = \Omega^{n-2}\mathcal{GT}_G^{n+2}.$$

(3)

$$\Sigma^{n-1}A = (n-1)\mathrm{Rep}(H) \in n\mathrm{Rep}(G) \hookrightarrow \mathfrak{Z}_1(n\mathrm{Rep}(G)).$$
$$\Sigma^n A = \mathsf{RMod}_{\Sigma^{n-1}A}(\mathfrak{Z}_1(n\mathrm{Rep}(G))) \in \Sigma\mathfrak{Z}_1(n\mathrm{Rep}(G)).$$

Translate $\Sigma^n A$ into an E_1-algebra in the second coordinate system of $\mathcal{GT}_G^{n+2}$. It is defined by the following monoidal functor:

$$n\mathrm{Rep}(G) \to n\mathrm{Rep}(G) \boxtimes_{\mathfrak{Z}_1(n\mathrm{Rep}(G))} \mathsf{RMod}_{\Sigma^{n-1}A}(\mathfrak{Z}_1(n\mathrm{Rep}(G)))$$
$$\simeq \mathsf{RMod}_{\Sigma^{n-1}A}(n\mathrm{Rep}(G)) \simeq n\mathrm{Rep}(H).$$

(4) By condensing $\Sigma^n A = n\mathrm{Rep}(H)$, we obtain $\mathsf{D}^{n+2} = \mathsf{GT}_H^{n+2}$ as the condensed phase:

$$\mathcal{D} \simeq \mathsf{Mod}^{\mathrm{E}_1}_{n\mathrm{Rep}(H)^{\mathrm{op}}}(\mathcal{GT}_G^{n+2}) \simeq \mathsf{Mod}^{\mathrm{E}_1}_{n\mathrm{Rep}(H)^{\mathrm{op}}}((n+1)\mathrm{Vec}) \simeq \mathcal{GT}_H^{n+2}.$$

As a consequence, we provides the precise mathematical theory behind the folklore that breaking the G-gauge "symmetry" in GT_G^{n+2} to a subgroup H gives the H-gauge theory GT_H^{n+2}.

Conclusion and outlooks

1. One can see that the theory of defect condensation is precisely a mathematical theory of higher representations of higher algebras or higher Morita theories. There will be a mathematical companion of this paper, in which we develop a mathematical theory of condensable E_k-algebras.
2. It is possible to developed a mathematical theory of condensations of gapless but liquid-like defects based on the theory of gapped/gapless quantum liquids K.-Zheng:1705.01087,1905.04924,1912.01760,2011.02859, which is a prehistorical theory of SymTO/SymTFT based on the so-called 'topological Wick rotation'.
3. Although a new paradigm is emerging, as far as I can tell, it is still far from being complete. A complete paradigm demands an entirely new calculus, in which we are still in the beginning stage to understand integers. It means that there are a lot of exciting problems to work on in the coming future.

Thank you!