A morphism between two QFT's

Liang Kong

August 16, 2023, "Categorical aspects of symmetries", Nordita, Stockholm, Sweden

SIQSE, Southern University of Science and Technology, China

References: K.-Wen-Zheng:1502.01690, 1702.00673, A morphism between two mathematical objects of the same type (e.g. groups, algebras, representations, categories, etc.), preserving the defining structures of the mathematical objects, is arguably the most important notion in mathematics. Ironically, such a notion for physical systems (e.g. QFT's) had never been introduced in physics.

A morphism between two mathematical objects of the same type (e.g. groups, algebras, representations, categories, etc.), preserving the defining structures of the mathematical objects, is arguably the most important notion in mathematics. Ironically, such a notion for physical systems (e.g. QFT's) had never been introduced in physics.

In 2015, we introduced the notion of a morphism (preserving the structures) between two topological orders (or quantum phases or QFT's) in K.-Wen-Zheng:1502.01690. In this talk, I will review this notion and its applications in the study of topological orders or quantum liquids K.-Zheng:1705.01087,1905.04924,1912.01760,2011.02859,2107.03858. In particular, we show that it naturally leads us to boundary-bulk relation and topological Wick rotation, which essentially equivalent to "categorical symmetry" or "Symmetry/TO correspondence" Ji-Wen:1912.13492, K.-Lan-Wen-Zheng-Zhang:2005.14178. We also show that the a morphism between QFT's is a more precise and fundamental structure underlying the idea of "topological symmetry" or "SymTFT" Freed-Moore-Teleman:2209.07471, Apruzzi-Bonetti-Etxebarria-Hosseini-Schäfer-Nameki:2112.02092.

Question: How to define a morphism between two QFT's?

Question: How to define a morphism between two QFT's?

A well known answer: A usual definition of a morphism between two QFT's is a domain wall between two QFT's. A domain wall provides a physical realization of the mathematical notion of a bimodule because the domain wall is naturally equipped with the two-side action of the operators in two QFT's.

Question: How to define a morphism between two QFT's?

A well known answer: A usual definition of a morphism between two QFT's is a domain wall between two QFT's. A domain wall provides a physical realization of the mathematical notion of a bimodule because the domain wall is naturally equipped with the two-side action of the operators in two QFT's.

Remark: However, such a definition of a morphism (as a bimodule) is less fundamental because it does not preserve the algebraic structures of a QFT. As a consequence, such a definition of a morphism (as a bimodule) only distinguishe QFT's up to Morita equivalences.

In mathematics, there is a more fundamental or natural notion of a homomorphism between two algebraic objects, i.e. a map preserving the algebraic structure.

1. A group homomorphism $f : G \to H$: $f(g_1g_2) = f(g_1)f(g_2)$ for $g_1, g_2 \in G$.
2. An algebra homomorphism $f : A \to B$ between two $\mathbb{C}$-linear algebras is a $\mathbb{C}$-linear map such that $f(ab) = f(a)f(b)$.

Question: What is the physical realization of the notion of an algebra homomorphism?

Or equivalently:

Question: How to map a quantum many-body system (or QFT) to another such that algebraic structures of operators or observables are preserved?

A morphism between QFT's or quantum liquids

Before we introduce this notion, we need some preparation. The term "topological order" in the following a few slides will be replaced eventually by (gapped/gapless) "quantum liquid", a notion which is, however, harder to define. Therefore, for convenience, we restrict our discussion to the familiar notion topological orders at first. We will come back to quantum liquids later.

Definition (K.-Wen:1405.5858)

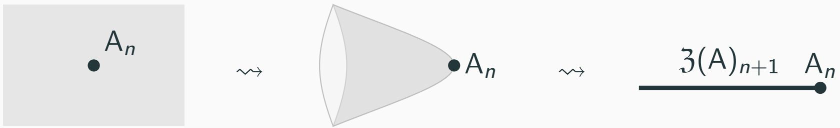
A topological order is called anomaly-free if it can be realized by a lattice model in the same dimension, and is called anomalous otherwise.

Examples: The gapped boundaries of non-trivial topological orders are examples of anomalous topological orders. A gapped boundary of the trivial n+1D topological order is an anomaly-free nD topological order.

Unique Bulk Hypothesis/Principle [K.-Wen:1405.5858]

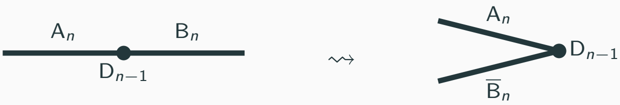
A (potentially anomalous) nD topological order A_n has a unique n+1D anomaly-free topological order as its *bulk*, denoted by $\mathfrak{Z}(\mathrm{A}_n) = \mathfrak{Z}(\mathrm{A})_{n+1}$.

Remark: An anomalous topological order must be realizable as a defect in a higher (but still finite) dimensional lattice model as illustrated below. Otherwise, it is safe to say that such a topological order does not exist. But such realizations are not unique.

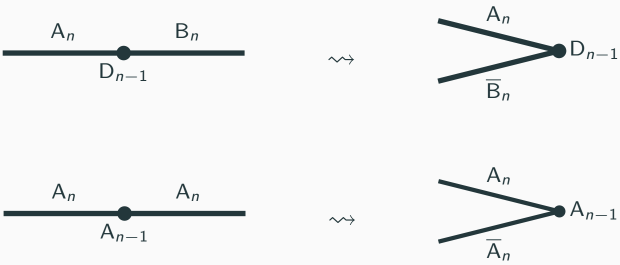


After the dimensional reduction, we obtain the unique *bulk* of A_n. Importantly, A_n should be understood as a boundary phase, which includes a neighborhood of the boundary by definition.

Example: If D_{n-1} is a domain wall between A_n and B_n, then $\mathfrak{Z}(\mathrm{D})_{n+1} = \mathrm{A}_n \boxtimes \overline{\mathrm{B}}_n$.



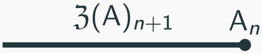
Example: If D_{n-1} is a domain wall between A_n and B_n, then $\mathfrak{Z}(\mathrm{D})_{n+1} = \mathrm{A}_n \boxtimes \overline{\mathrm{B}}_n$.



where A_{n-1} denotes the A_n restricting to the trivial domain wall of codimension 1.

Example: $\mathbb{1}_n$ = the trivial nD topological order. We have

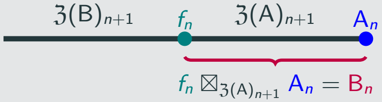
- $\mathfrak{Z}(\mathbb{1})_{n+1} = \mathbb{1}_{n+1}$;
- A_n is anomaly-free if and only if $\mathfrak{Z}(A)_{n+1} = \mathbb{1}_{n+1}$;
- $\mathfrak{Z}(\mathfrak{Z}(A))_{n+2} = \mathbb{1}_{n+2}$.



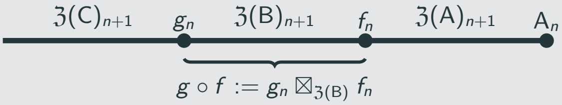
Remark: The statement of "the ***bulk*** of a ***bulk*** is trivial" is somewhat dual to that of "the boundary of a boundary is empty", which inspired homology theory. Therefore, we expect that "the ***bulk*** of a ***bulk*** is trivial" should lead us to a non-trivial "cohomology theory".

Definition (K.-Wen-Zheng:1502.01690, 1702.00673)

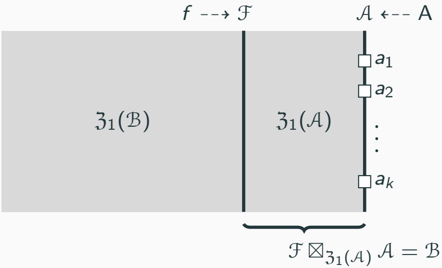
A morphism $f : A_n \to B_n$ between two topological orders A_n and B_n (both having gapped bulks) is a gapped wall f_n between $\mathfrak{Z}(A)$ and $\mathfrak{Z}(B)$ such that



The composition of two morphisms $f : A_n \to B_n$ and $g : B_n \to C_n$ is defined as follows:



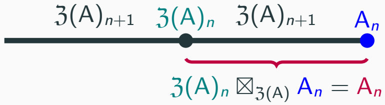
This definition of a morphism
coincides with the mathematical notion of
a monoidal functor between two fusion n-categories



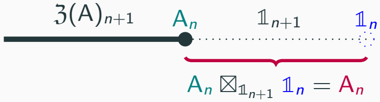
$$\begin{array}{rcl} \text{monoidal functors} & \Longleftrightarrow & \text{domain walls between } \mathfrak{Z}_1(\mathcal{B}) \text{ and } \mathfrak{Z}_1(\mathcal{A}) \\ (\mathcal{A} \xrightarrow{f} \mathcal{B}) & \longmapsto & \mathcal{F} := \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}({}_f\mathcal{B}, {}_f\mathcal{B}), \\ \left(\mathcal{A} \xrightarrow{f} \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} = \mathcal{B}\right) & \longleftarrow & \mathcal{F} \end{array}$$

These two constructions are inverse of each other. K.-Zheng:1507.00503, 2107.03858

Examples 1: $A_n \xrightarrow{\mathrm{id}_A} A_n$ is defined by the trivial domain wall $\mathfrak{Z}(A)_n$ in $\mathfrak{Z}(A)_{n+1}$.



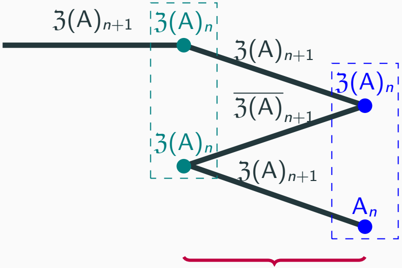
Examples 2: Let $\mathbb{1}_n$ be the trivial nD topological order. There is a canonical morphism $\iota_A : \mathbb{1}_n \to A_n$ defined by:



Examples 3: There is a canonical morphism

$$\mathfrak{Z}(A)_n \boxtimes A_n \xrightarrow{m} A_n$$

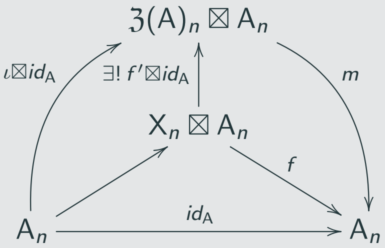
defined as follows



$$(\mathfrak{Z}(A)_n \boxtimes \mathfrak{Z}(A)_n) \boxtimes_{\mathfrak{Z}(A)\boxtimes\overline{\mathfrak{Z}(A)}\boxtimes\mathfrak{Z}(A)} (\mathfrak{Z}(A)_n \boxtimes A_n) = A_n$$

Theorem (K.-Wen-Zheng:1502.01690, 1702.00673)

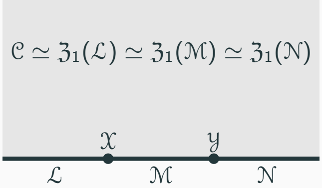
The pair $(\mathfrak{Z}(\mathrm{A})_n, m)$ satisfies the universal property of center. That is, if X is an nD topological order equipped with a morphism $f : \mathrm{X}_n \boxtimes \mathrm{A}_n \to \mathrm{A}_n$, then there is a unique morphism $f' : \mathrm{X}_n \to \mathfrak{Z}(\mathrm{A})_n$ such that the following diagram is commutative:



Remark: This theorem simply says that the *bulk* is the center of the boundary.

This universal property works for all the well-known notions of centers.

1. When A is a group and m is a group homomorphism, it recovers the center of a group $Z(\mathrm{A}) = \{z \in \mathrm{A} | zg = gz, \forall g \in \mathrm{A}\}$.
2. When A is an algebra and m is an algebraic homomorphism, it recovers the usual center of an algebra $Z(\mathrm{A}) = \{z \in \mathrm{A} | za = az, \forall a \in \mathrm{A}\}$.
3. When A is a fusion category and m is a monoidal functor, it recovers the Drinfeld center.
4. When A is a braided fusion category and m is a braided monoidal functor, it recovers the Müger center.
5. The center of open-string CFT is the closed CFT and is called a full center. Fjelstad-Fuchs-Runkel-Schweigert:math.CT/0512076, K.-Runkel:0708.1897, Davydov:0908.1250
6. Generalized Deligne Conjecture (Kontsevich): the E_n-center of an E_n-algebra is an E_{n+1}-algebra. Lurie's book "Higher Algebras"

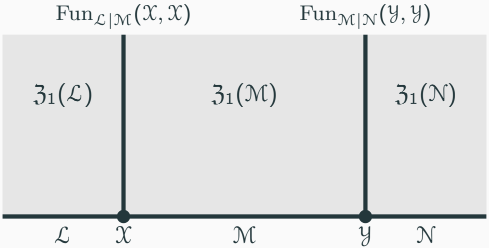


For 2+1D non-chiral topological orders, we have recovered the well-known result. In this case, a 3D topological order can be described by a modular tensor category (MTC) $\mathcal{C}$ Moore-Seiberg:1989, Kitaev:cond-mat/0506438 and its gapped boundaries can be described by fusion categories $\mathcal{L}, \mathcal{M}, \mathcal{N}$ Kitaev-K.:1104.5047. Moreover, we have

1. $\mathcal{C} \simeq \mathfrak{Z}_1(\mathcal{L}) \simeq \mathfrak{Z}_1(\mathcal{M}) \simeq \mathfrak{Z}_1(\mathcal{N})$ Kitaev-K.:1104.5047, Fuchs-Schweigert-Valentino:1203.4568, K.:1307.8244
2. $\mathcal{X}$ is an invertible $\mathcal{L}$-$\mathcal{M}$-bimodule that defines an Morita equivalence between $\mathcal{L}$ and $\mathcal{M}$.

Remark: Mathematically, two fusion categories are Morita equivalent if and only if they share the same center. Etingof-Nikshych-Ostrik:0809.3031.

$\mathcal{L}, \mathcal{M}, \mathcal{N}$ are fusion n-categories.
${}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}}, {}_{\mathcal{M}}\mathcal{Y}_{\mathcal{N}}$ are bimodules.



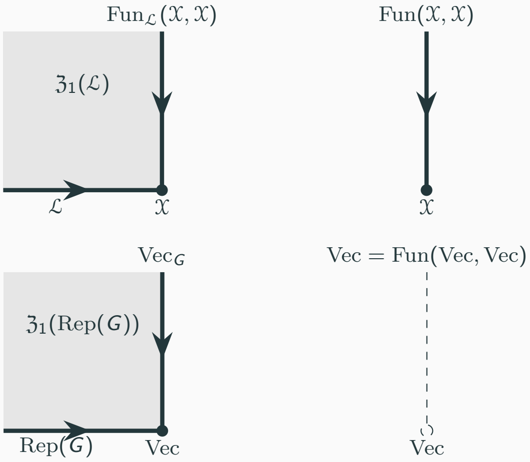
Theorem (Boundary-Bulk Relation with Defects, K.-Zheng:1507.00503, 2107.03858)

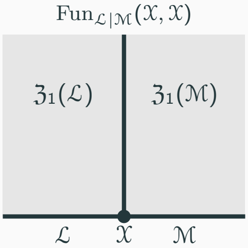
The assignment $\mathcal{L} \mapsto \mathfrak{Z}_1(\mathcal{L}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{L}}(\mathcal{L}, \mathcal{L})$ *and* $\mathcal{X} \mapsto \mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X}, \mathcal{X})$ *defines a functor.*

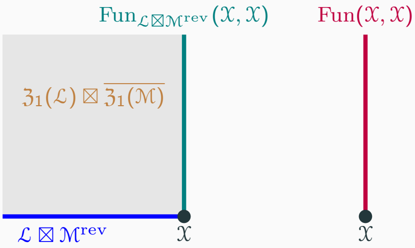
$$\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X}, \mathcal{X}) \boxtimes_{\mathfrak{Z}_1(\mathcal{M})} \mathrm{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{Y}, \mathcal{Y}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{N}}(\mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{Y}, \mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{Y})$$

Fundamental formula in computing a fusion or dimensional reduction:

$$\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathfrak{Z}_1(\mathcal{M})} \mathrm{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{Y},\mathcal{Y}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{N}}(\mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y}, \mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y})$$

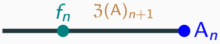






Theorem[K.-Zheng:1507.00503]: When $n=1$, this center functor is a monoidal equivalence.

Proof : $\quad {}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}} = {}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}\mathcal{X} \iff$ a monodial functor : $\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}} \to \mathrm{Fun}(\mathcal{X},\mathcal{X})$

Recall : $(f : A \to B) \quad =$ 

$\iff$ a domain wall : $\mathrm{Fun}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}(\mathcal{X},\mathcal{X}) = \mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X})$.

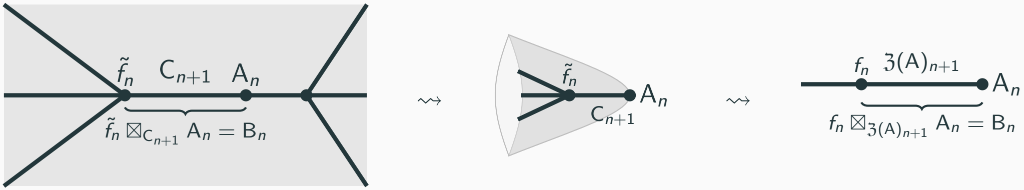
It automatically includes (1) $\mathcal{A} \sim^{\mathrm{Morita}} \mathcal{B}$ iff $\mathfrak{Z}_1(\mathcal{A}) \simeq \mathfrak{Z}_1(\mathcal{B})$ Etingof-Nikshych-Ostrik:0809.3031; (2) $\mathrm{BrPic}(\mathcal{A}) \simeq \mathrm{Aut}^{br}(\mathfrak{Z}_1(\mathcal{A}))$ Etingof-Nikshych-Ostrik:0909.3140. For $n>1$, it is not an equivalence.

The formal proof of boundary-bulk relation assumed only the well-definedness of the notion of a morphism between two quantum phases, which is further based on the uniqueness of the bulk and the well-definedness of the fusion of domain walls. This notion and the formal proof should also work for certain 'nice' gapless phases.

1. For 1+1D rational CFT's, this boundary-bulk relation reproduces the so-called open-closed duality for 1+1D RCFT's, i.e. the bulk (closed) CFT is the center of a boundary (or open) CFT, a result which was known long ago.
   Fjelstad-Fuchs-Runkel-Schweigert:math.CT/0512076, K.-Runkel:0708.1897, Davydov:0908.1250

Definition: Those (gapped/gapless) quantum phases such that above formal proof of boundary-bulk relation works will be called "quantum liquids", a notion which can be more precisely defined as a 'fully dualizable QFT' K.-Zheng:2011.02859

Remark: Only a few days ago, I noticed that Grady-Pavlov had introduced a notion called 'fully dualizable geometric QFT' in Grady-Pavlov:2111.01095.



Note that the pair $(\tilde{f}_n, C_{n+1})$ also realizes physically a morphism $f : A_n \to B_n$. But such physical realizations of the same morphism are not unique in general.

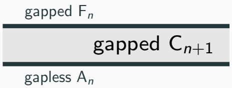
Such physical realizations of a morphism, although non-unique, are still very useful and was called a 'weak morphism' in Appendix A.3 in K.-Wen-Zheng:1502.01690.

Remark: This notion seems to work for all (quantum) many-body systems if we choose a scheme of crossgraining in order to make sense of fusion and abandon the (most strict) associativity of the composition, which should not hold for generic non-topological systems.

Question to all physicists: A morphism for classical systems?

Relation to the idea of topological symmetry or SymTFT: Freed-Moore-Teleman:2209.07471, Apruzzi-Bonetti-Etxebarria-Hosseini-Schäfer-Nameki:2112.02092

$$\mathrm{B}_n := \mathrm{F}_n \boxtimes_{\mathrm{C}_{n+1}} \mathrm{A}_n := \Bigg\{$$



(1) For us, F_n defines a morphism $\mathrm{A}_n \to \mathrm{B}_n$ and A_n defines a morphism $\mathrm{F}_n \to \mathrm{B}_n$.

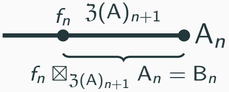
(2) The intuition that the pair (F, C) acts on B can be stated more precisely.

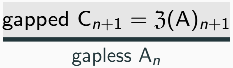
- $(\mathrm{F}_n \boxtimes \mathrm{C}_n) \boxtimes \mathrm{B}_n \to \mathrm{F}_n \boxtimes_{\mathrm{C}_{n+1}} \mathrm{C}_n \boxtimes_{\mathrm{C}_{n+1}} \mathrm{A}_n \boxtimes \mathrm{B}_n \simeq \mathrm{B}_n \boxtimes \mathrm{B}_n \to \mathrm{B}_n$. This "$(\mathrm{F}, \mathrm{C})$-action on B" does not preserves the algebraic structure of B. Hence, it is not an action in usual sense.
- Topological defects in F_n is mapped into those in B_n, the latter of which are non-invertible symmetries thus "acting" on B_n (preserving dynamics). This "action" is not compatible with the view that both dynamics and topological defects are defining data of B_n.
- Can we define an action $\mathrm{X}_n \boxtimes \mathrm{A}_n \to \mathrm{A}_n$ that preserving all algebraic structure of A_n? Yes, by the universal property of the center, such an X_n-action on A_n must factor through the canonical $\mathfrak{Z}(\mathrm{A})_n$-action on A_n, i.e. $\mathfrak{Z}(\mathrm{A})_n \boxtimes \mathrm{A}_n \xrightarrow{m} \mathrm{A}_n$.

Gapless boundaries of 2+1D topological orders

Consequences of boundary-bulk relation:

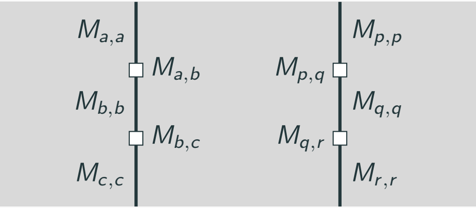
1. It leads to the classification of all n+1D anomaly-free topological orders (up to invertible ones) by fusion n-categories with a trivial E_1-center or braided fusion n-categories with a trivial E_2-center; K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, Johnson-Freyd:2003.06663
2. The proof applies to an n+1D topological order C_{n+1} with a gapless boundary A_n.





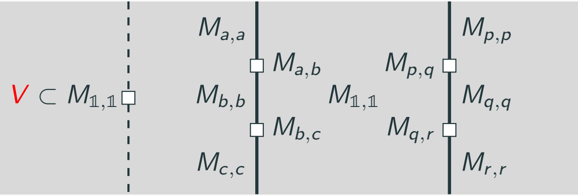
Since we already have a precise categorical description of C_{n+1} as a braided fusion n-categories $\mathcal{C}_n$ with a trivial E_2-center, then we can find the categorical description of a gapless boundary A_n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$.

Consider the 1+1D worldsheet of a gapless boundary of the 2+1D topological order $(\mathcal{C}, c)$.



It turns out that the macroscopic observables on the 1+1D worldsheet form an enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ K.-Zheng:1705.01087, 1905.04924, 2011.02859, K.-Wen-Zheng:2108.08835, where

1. $a, b, c \in \mathcal{S}$ are the labels of topological defect lines (TDL);
2. $M_{a,b}$ is the spaces of fields living on the 0D defect junction; in particular, it means that the space of fields living on the TDL label by 'a' is just $M_{a,a}$;
3. OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ of defect fields defines a kind of 'composition map' such that all $\{M_{a,b}\}_{a,b\in\mathcal{S}}$ together form a structure similar to that of a category $(\{\hom(a,b)\}_{a,b\in\mathcal{C}})$.
4. We will show next that $M_{a,b}, \circ \in \mathcal{B}$ and we obtain a $\mathcal{B}$-enriched category ${}^{\mathcal{B}}\mathcal{S}$.



Let $\mathbb{1} \in \mathcal{S}$ be the label of the trivial TDL. Then fields in $M_{\mathbb{1},\mathbb{1}}$ can live in the 2-cell. A subalgebra of fields generated by $\langle T(z,\bar{z})\rangle \subset M_{\mathbb{1},\mathbb{1}}$ is transparent to all TDL's. Without loss of generality, we assume that $\langle T(z,\bar{z})\rangle \subset V \subset M_{\mathbb{1},\mathbb{1}}$ is transparent to all TDL's. Assume V is rational, i.e. Mod_V is a MTC. Moore-Seiberg:1989, Huang:math/0502533

$M_{a,b}$ is clearly a V-module (with a 2-dimensional V-action), i.e. $M_{a,b} \in \mathrm{Mod}_V$. The compatibility between the OPE $M_{b,c} \otimes_{\mathbb{C}} M_{a,b} \xrightarrow{\circ} M_{a,c}$ and the V-action is equivalent to a morphism $M_{b,c} \otimes_V M_{a,b} \xrightarrow{\circ} M_{a,c}$ in Mod_V. As a consequence, we obtain an Mod_V-enriched fusion category ${}^{\mathrm{Mod}_V}\mathcal{S}$, where $\hom_{{}^{\mathrm{Mod}_V}\mathcal{S}}(a,b) = M_{a,b}$ and $\otimes$ is the horizontal fusion of TDLs. It turns out that all correlation functions and the OPE among defect fields can be recovered from $(V, {}^{\mathrm{Mod}_V}\mathcal{S})$.Huang:math/0303049,math/0502533, Fuchs-Runkel-Schweigert:2001-2006, Huang-Kirillov-Lepowsky:1406.3420, K.:0807.3356, Davydov-K.-Runkel:1307.5956

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

The 'rational' gapped/gapless boundaries of a 2+1D topological order $(\mathcal{C}, c)$ can be completely characterized or classified by the triples $(V, \phi, \mathcal{S})$ or $(V, \phi, {}^{\mathcal{B}}\mathcal{S})$, where

1. *for a chiral gapless boundary, V is a rational VOA of central charge c;* Huang:math/0502533
   for a non-chiral gapless boundary, V is a rational full field algebra $(c_L - c_R = c)$ K.-Huang:math/0511328*; when $V = \mathbb{C}$, it describes a gapped boundary.* Kitaev-K.:1104.5047
2. *$\mathcal{S}$ is a fusion category equipped with a braided equivalence $\phi : \mathfrak{C} \boxtimes \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathfrak{Z}_1(\mathcal{S})$.*

Moreover, the restriction $\phi : \overline{\mathrm{Mod}_V} \xrightarrow{\simeq} \mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})} = \overline{\mathrm{Mod}_V}$ is a braided equivalence, which determines a $\mathcal{B}$-enriched fusion category ${}^{\mathcal{B}}\mathcal{S}$ via the so-called canonical construction, i.e. $M_{a,b} = [a, b] \in \mathcal{B}$. Lindner:1981,Morrison-Penneys:1701.00567

Theorem (K.-Zheng:1704.01447, K.-Yuan-Zhang-Zheng:2104.03121)

The bulk is the center of a boundary, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$.

Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a pair* $\mathsf{X} = (\mathsf{X}_{\rm lqs}, \mathsf{X}_{\rm top})$*, where*

1. $\mathsf{X}_{\rm lqs} = (V, \phi)$ *is called* *local quantum symmetry* *(containing dynamical information);*
2. $\mathsf{X}_{\rm top} = {}^{\mathcal{B}}\mathcal{S}$ *is called* *topological skeleton* *(recall* $\mathcal{B} := \mathcal{C}'_{\mathfrak{Z}_1(\mathcal{S})}$*).*

Moreover, $\mathsf{X}_{\rm top}$ *can be obtained by topological Wick rotation and* $\mathfrak{Z}_1(\mathsf{X}_{\rm top}) \simeq \mathcal{C}$*.*

When $(\mathcal{C}, c)$ *is trivial, we obtain a holographic duality between a 3D theory and a 2D theory.*

Examples of gapped boundaries and chiral gapless boundaries: (skip unless people ask questions)

1. For a non-chiral 2+1D topological order $(\mathfrak{Z}(\mathcal{A}), 0)$, where $\mathcal{A}$ is a fusion category, the triple $(\mathbb{C}, \phi, {}^{\mathrm{Vec}}\mathcal{A} = \mathcal{A})$, where $\mathrm{Mod}_{\mathbb{C}} \xrightarrow{\phi=\mathrm{id}} \mathrm{Vec}$, defines a gapped boundary.
2. For the E_8 invertible 2+1D topological order $(\mathrm{Vec}, 8)$, the triple $(V_{E_8}, \phi, {}^{\mathrm{Vec}}\mathrm{Vec})$, where $\mathrm{Mod}_{V_{E_8}} \xrightarrow{\phi=\mathrm{id}} \mathrm{Vec}$, defines a non-trivial gapless boundary.
3. Let V be a rational VOA and $\mathcal{C} = \mathrm{Mod}_V$ is MTC. The triple $(V, \mathrm{id}, {}^{\mathcal{C}}\mathcal{C})$ defines a gapless boundary of $(\mathcal{C}, c)$ and $\mathfrak{Z}({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$.

Examples of non-chiral gapless edges: (skip unless people ask questions)

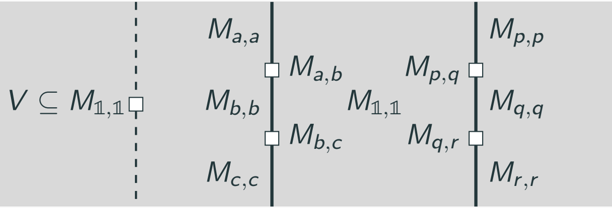
- Three modular tensor categories (MTC):
  1. $\mathrm{Is} := \mathrm{Mod}_{V_{\mathrm{Is}}}$ where V_{Is} is the Ising VOA with the central charge $c = \frac{1}{2}$. It has three simple objects $\mathbb{1}, \psi, \sigma$ (i.e. $\mathbb{1} = V_{\mathrm{Is}}$) and the following fusion rules:
  $$\psi \otimes \psi = \mathbb{1}, \quad \psi \otimes \sigma = \sigma, \quad \sigma \otimes \sigma = \mathbb{1} \oplus \psi.$$
  2. $\mathfrak{Z}_1(\mathrm{Is}) \simeq \mathrm{Is} \boxtimes \overline{\mathrm{Is}}$. It has 9 simple objects: $\mathbb{1} \boxtimes \mathbb{1}, \mathbb{1} \boxtimes \psi, \mathbb{1} \boxtimes \sigma, \psi \boxtimes \mathbb{1}, \cdots$.
  3. TC $=$ the MTC of toric code. It has four simple objects $1, e, m, f$ and the following fusion rules:
  $$e \otimes e = m \otimes m = f \otimes f = 1, \quad m \otimes e = f.$$

- Three non-chiral symmetries (i.e. full field algebras Huang-K.:math/0511328):
  1. $P = V_{\mathrm{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathrm{Is}}} = \mathbb{1} \boxtimes \mathbb{1} \in \mathrm{Is} \boxtimes \overline{\mathrm{Is}} = \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_P = |\chi_0|^2$;
  2. $Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \in \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_Q = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2$
  3. $R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma \in \mathfrak{Z}_1(\mathrm{Is})$ $\Rightarrow \chi_R = |\chi_0|^2 + |\chi_{\frac{1}{2}}|^2 + |\chi_{\frac{1}{16}}|^2$

  We have $P \lneq Q \lneq R$.

- P, Q, R are condensable algebras in $\mathfrak{Z}_1(\mathrm{Is})$ and R is a Lagrangian algebra.
  1. $\phi_P : \mathrm{Mod}_P = (\mathfrak{Z}_1(\mathrm{Is}))_P^0 \xrightarrow{\simeq} \mathfrak{Z}_1(\mathrm{Is})$: condensing P gives $\mathfrak{Z}_1(\mathrm{Is})$;
  2. $\phi_Q : \mathrm{Mod}_Q = (\mathfrak{Z}_1(\mathrm{Is}))_Q^0 \xrightarrow{\simeq} \mathrm{TC}$: condensing Q in $\mathfrak{Z}_1(\mathrm{Is})$ gives toric code TC Bais-Slingerland:0808.0627, Chen-Jian-K.-You-Zheng:1903.12334;
  3. $\phi_R : \mathrm{Mod}_R = (\mathfrak{Z}_1(\mathrm{Is}))_R^0 \xrightarrow{\simeq} \mathrm{Vec}$: condensing R in $\mathfrak{Z}_1(\mathrm{Is})$ gives the trivial phase.

$$P = V_{\mathsf{Is}} \otimes_{\mathbb{C}} \overline{V_{\mathsf{Is}}} = \mathbb{1} \boxtimes \mathbb{1}$$
$$Q = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi$$
$$R = \mathbb{1} \boxtimes \mathbb{1} \oplus \psi \boxtimes \psi \oplus \sigma \boxtimes \sigma$$
$$\phi_P : \mathrm{Mod}_P \xrightarrow{\simeq} \mathfrak{Z}_1(\mathsf{Is})$$
$$\phi_Q : \mathrm{Mod}_Q \xrightarrow{\simeq} \mathsf{TC}$$
$$\phi_R : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$$



- Four anomaly-free 1+1D gapless quantum liquids defined by triples: i.e. its 2+1D bulk topological order is trivial: $(\mathcal{C}, c) = (\mathrm{Vec}, 0)$ and $\mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S}) \simeq \mathrm{Vec}$.
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathsf{Is})$: in this case $V = P \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Rep}(\mathbb{Z}_2))$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  3. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathrm{Vec}_{\mathbb{Z}_2})$: in this case $V = Q \subsetneq R = M_{\mathbb{1},\mathbb{1}}$;
  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})$: in this case $V = R = M_{\mathbb{1},\mathbb{1}}$.

  In all 4 cases, the space of non-chiral fields living on each 2-cells (i.e. $M_{\mathbb{1},\mathbb{1}}$) is given by the same modular-invariant closed CFT R.

- Gappable gapless edges of 2+1D toric code: TC $= \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$,
  Chen-Jian-K.-You-Zheng:arXiv:1903.12334, K.-Zheng:1912.01760
  1. $(P, \phi_P, {}^{\mathfrak{Z}_1(\mathsf{Is})}\mathcal{S})$, where $\mathcal{S} = (\mathfrak{Z}_1(\mathsf{Is}))_Q$ is the fusion category of the right Q-modules in $\mathfrak{Z}_1(\mathsf{Is})$. $\mathcal{S}$ has 6 simple objects $\mathbb{1}, e, m, f, \chi_\pm$, where $\mathbb{1}, e, m, f$ can be identified with 4 anyons in the bulk and $\chi_\pm$ can be identified with two twist defects in the bulk.
  2. $(Q, \phi_Q, {}^{\mathsf{TC}}\mathsf{TC})$ = the canoncial gapless edge;
  3. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Rep}(\mathbb{Z}_2) = \mathrm{Rep}(\mathbb{Z}_2))$, where $\phi : \mathrm{Mod}_R \xrightarrow{\simeq} \mathrm{Vec}$. Moreover, we have

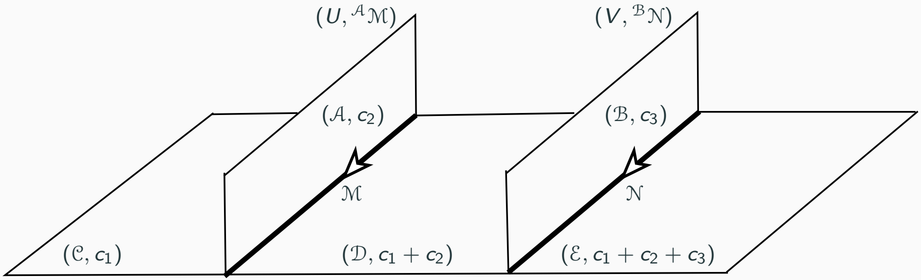
$$(R, \phi_R, \mathrm{Rep}(\mathbb{Z}_2)) = \underbrace{(\mathbb{C}, \mathrm{id}, \mathrm{Rep}(\mathbb{Z}_2))}_{\text{the smooth gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}.$$

  4. $(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec}_{\mathbb{Z}_2})$:

$$(R, \phi_R, \mathrm{Vec}_{\mathbb{Z}_2}) = \underbrace{(\mathbb{C}, \mathrm{id}, \mathrm{Vec}_{\mathbb{Z}_2})}_{\text{the rough gapped edge}} \boxtimes \underbrace{(R, \phi_R, {}^{\mathrm{Vec}}\mathrm{Vec})}_{\text{an anomaly-free 2D gapless liquid}}$$

  5. (smooth/rough gapped edge) $\boxtimes$ (any anomaly-free 2D gapless liquid).

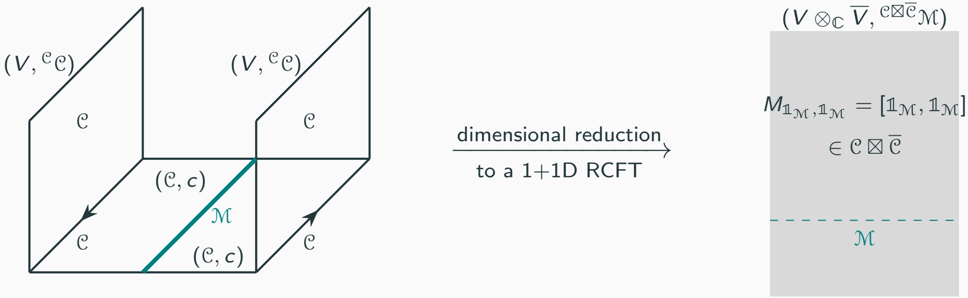
How to compute the fusion of two gapless domain walls?



K.-Zheng:1705.01087 $$(U, {}^{\mathcal{A}}\mathcal{M}) \boxtimes_{(\mathcal{D}, c_1+c_2)} (V, {}^{\mathcal{B}}\mathcal{N}) = (U \otimes_{\mathbb{C}} V, {}^{\mathcal{A}\boxtimes\mathcal{B}}\mathcal{M} \boxtimes_{\mathcal{D}} \mathcal{N}),$$

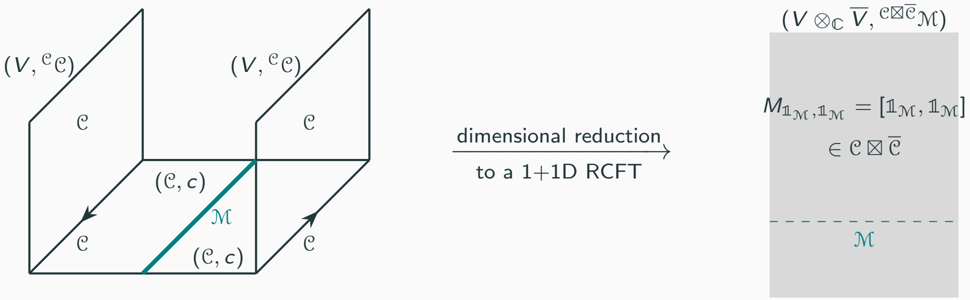
where U, V are VOAs or full field algebras.

Consider a 2+1D topological order $(\mathcal{C}, c)$ with the canonical gapless boundary $(V, {}^{\mathcal{C}}\mathcal{C})$, where V is a VOA and $\mathcal{C} = \mathrm{Mod}_V$, and a gapped domain wall $\mathcal{M}$: K.-Zheng:1705.01087



$$(V, {}^{\mathcal{C}}\mathcal{C}) \boxtimes_{(\mathcal{C},c)} (\mathbb{C}, {}^{\mathrm{Vec}}\mathcal{M}) \boxtimes_{(\mathcal{C},c)} (\overline{V}, {}^{\overline{\mathcal{C}}}\mathcal{C}^{\mathrm{rev}}) = (V \otimes_{\mathbb{C}} \overline{V}, {}^{\mathcal{C}\boxtimes\overline{\mathcal{C}}}\mathcal{M}).$$

$M_{\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}} = [\mathbb{1}_{\mathcal{M}}, \mathbb{1}_{\mathcal{M}}] \in \mathcal{C} \boxtimes \overline{\mathcal{C}}$ recovers all modular invariant 1+1D rational CFT's.



$\{$modular invariant CFT's extending $V \otimes_{\mathbb{C}} \overline{V}\}$

$\longleftrightarrow$ $\{$Lagrangian algebras in $\mathcal{C} \boxtimes \overline{\mathcal{C}}\}$ K.-Runkel:0807.3356, Müger:0909.2537

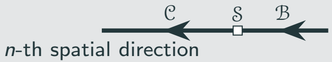
$\longleftrightarrow$ $\{$gapped boundaries $\mathcal{M}$ of $(\mathcal{C} \boxtimes \overline{\mathcal{C}}, 0)\}$ K.:1307.8244

$\longleftrightarrow$ $\{$gapped domain walls $\mathcal{M}$ in $(\mathcal{C}, c)\}$ folding trick

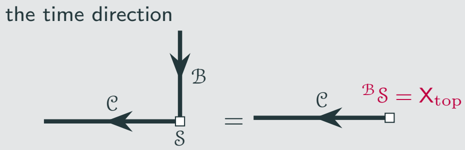
Topological Wick rotation, categorical symmetry and SymTFT

Topological Wick rotation in all dimensions: K.-Zheng:1905.04924,1912.01760,2011.02859

For a (potentially anomalous) quantum liquid $X = (X_{lqs}, X_{top})$, its topological skeleton X_{top} can be obtained by topological Wick rotation.



$\mathcal{S}$ is an "anomalous gapped boundaries" of an n+1D topological order $\mathcal{C}$



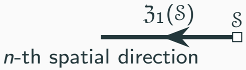
*n*D potentially gapless boundaries of an n+1D topological order $\mathcal{C}$

The boundary-bulk relation holds, i.e. $\mathcal{C} \simeq \mathfrak{Z}_1({}^{\mathcal{B}}\mathcal{S})$ K.-Zheng:in preparation. A mathematical theory of X_{lqs}, based on a theory of topological nets of (symmetric) local operator algebras in *n*D generalizing that of conformal nets in 2D, was developed K.-Zheng:2201.05726

Remark: In many examples, the same X_{top} can be realized by both gapped and gapless phases.

A new type of holographic dualities based on the idea of Topological Wick Rotation

K.-Zheng:1705.01087, 1905.04924, 1912.01760, 2011.02859: (nD is the spacetime dimension.)



an $n+1$D topological order with a gapped boundary	an nD quantum liquid (SPT/SET/SSB/gapless) with an internal symmetry of finite type
$\mathcal{S}$ is the category of topological defects on the boundary	$\mathcal{S}$ is the category of topological defects { the superselection (charge) sectors of states }
$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological defects in the bulk	$\mathfrak{Z}_1(\mathcal{S})$ is the category of topological sectors of operators { the spaces of non-local operators invariant under LOA}
$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$	$\mathfrak{Z}_1(\mathcal{S})$ naturally acts on $\mathcal{S}$

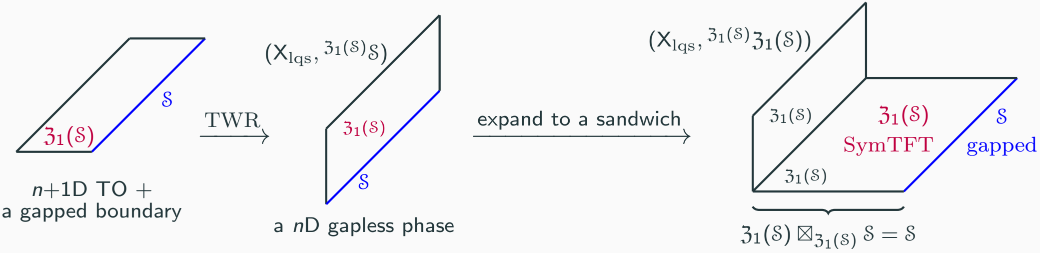
This holographic duality (between $n+1$D and nD) based on the idea of topological Wick rotation is closely related to that of "categorical symmetry" or "Symm/TO correspondence".

1. The notion of categorical symmetry (i.e. $\mathfrak{Z}_1(\mathcal{S})$) was proposed in Ji-Wen:1912.13492 in an attempt to combine a symmetry G with its dual symmetry in the study of critical phenomena. Importantly, it can be explicitly constructed from "patch charge operators" in nD (parallel to the Mod_V in K.-Zheng:1705.01087 and the "topological sectors of operators" in K.-Wen-Zheng:2108.08835).
2. It was more systematically developed in K.-Lan-Wen-Zheng-Zhang:2003.08898,2005.14178, together with the classification of gapped quantum liquids with finite internal symmetries. This classification result is the same as the one obtained via topological Wick rotation K.-Zheng:2011.02859 but based on very different ideas.
3. Patch charge operators was further developed in Chatterjee-Wen:2203.03596
4. Both the idea of topological Wick rotation and Symm/TO correspondence were used in the study of topological phase transitions in 1+1D. Chen-Jian-Kong-You-Zheng:1903.12334, K.-Zheng:1912.01760,Ji-Wen:1912.13492,Chatterjee-Wen:2205.06244,Lu-Yang:2208.01572,Chatterjee-Ji-Wen:2212.14432

This holographic dualities were explicitly/implicitly discovered and further studied by different groups of people in different contexts. An incomplete list:

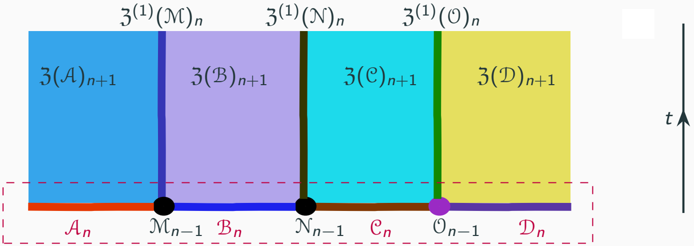
- Generalized Kramers-Wannier dualities: Freed-Teleman:1806.00008, Lootens-Delcamp-Ortiz-Verstraete:2112.09091.
- Categorical Symmetries: Ji-Wen:1912.13492, K.-Lan-Wen-Zhang-Zheng:2003.08898,2005.14178, Albert-Aasen-Xu-Ji-Alicea-Preskill:2111.12096, Chatterjee-Wen:2203.03596,2205.06244, Liu-Ji:2208.09101, Chatterjee-Ji-Wen:2212.14432
- Topological Wick Rotation: K.-Zheng:1705.01087, 1905.04924, 1912.01760, K.-Zheng:2011.02859, K-Wen-Zheng:2108.08835, Xu-Zhang:2205.09656, Lu-Yang:2208.01572
- Classical Statistical Models: Aasen-Mong-Fendley:1601.07185, 2008.08598
- SymTFT: Gaiotto-Kulp:2008.05960, Bhardwaj-Lee-Tachikawa:2009.10099, Apruzzi-Bonetti-Etxebarria-Hosseini-Schafer-Nameki:2112.02092, Freed-Moore-Teleman:2209.07471, Apruzzi:2203.10063, Moradi-Moosavian-Tiwari:2207.10712, ...
- Strange correlators: Bal-Williamson-Vanhove-Bultinck-Haegeman-Verstraete:1801.05959

Relation to the idea of SymTFT: Expand the system to a sandwich such that the fusion category symmetry $\mathcal{S}$ lives on one side of the sandwich and leave the dynamical data on the other side.



1. Our theory gives a more precise explanation of the ideas in “SymTFT”.
2. It also tells you that you do not need to “expand it to a sandwich” because the graphic notion for a gapless phase obtained from TWR is already ‘the half of a sandwich’ or a ‘quiche’ named in Freed-Moore-Teleman:2209.07471.

Categories of quantum liquids



We denote the category of nD anomaly-free quantum liquids by QL^n (morphisms are domain walls) and that of the topological skeletons of nD quantum liquids by $\mathrm{QL}^n_{\mathrm{top}}$.

Theorem (K.-Zheng:2011.02859, 2201.05726)

$$\mathrm{QL}^n \simeq \mathrm{QL}^n_{\mathrm{top}} \simeq \bullet/(n+1)\mathrm{Vec},$$

where $(n+1)\mathrm{Vec} = \Sigma^n\mathrm{Vec} = \Sigma^{n+1}\mathbb{C}$ is the category of nD non-chiral topological orders (that admit gapped boundaries) and higher codimensional defects. *Gaiotto-Johnson-Freyd:1905.09566.*

Summary: Main goal of this talk is to promote the notion of a morphism between QFT's, and to show that it is useful and powerful. Indeed, it alone had led us to the formal proof of boundary-bulk relation, to the discovery of topological Wick rotation, and to a unified mathematical theory of gapped and gapless quantum liquids, and to the study of the categories of quantum liquids.

Similar to the fact that category theory once revolutionized algebraic geometry by Grothendieck and his school, we believe that it will also revolutionize the theories of QFTs, phase transitions and perhaps quantum gravity. The important thing is not only to borrow categorical language for physical use, but also to use the spirit of the category theory to ask new questions and find new truths. To define the notion of a morphism between QFT's is only an example of many possibilities.

If the notion of a morphism is arguably the most important concept in mathematics, it is only reasonable that the notion of a morphism between QFT's is an important concept in physics.

Thank you !