A morphism between two QFT's

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References: K.-Wen-Zheng:1502.01690, 1702.00673, A morphism between two mathematical objects of the same type (e.g. groups, algebras, representations, categories, etc.), preserving the defining structures of the mathematical objects, is arguably the most important notion in mathematics. Ironically, such a notion for physical systems (e.g. QFT's) had never been introduced in physics.

A morphism between two mathematical objects of the same type (e.g. groups, algebras, representations, categories, etc.), preserving the defining structures of the mathematical objects, is arguably the most important notion in mathematics. Ironically, such a notion for physical systems (e.g. QFT's) had never been introduced in physics.

In 2015, we introduced the notion of a morphism (preserving the structures) between two topological orders (or quantum phases or QFT's) in K.-Wen-Zheng:1502.01690. In this talk, I will review this notion and some of its applications in the study of topological orders or quantum liquids K.-Zheng:1705.01087,1905.04924,1912.01760,2011.02859,2107.03858.

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Remark: However, such a definition of a morphism (as a bimodule) is less fundamental because it does not preserve the algebraic structures of a QFT. As a consequence, such a definition of a morphism (as a bimodule) only distinguish QFT's up to Morita equivalences.

In mathematics, there is a more fundamental or natural notion of a homomorphism between two algebraic objects, i.e. a map preserving the algebraic structure.

1. A group homomorphism $f : G \to H$: $f(g_1g_2) = f(g_1)f(g_2)$ for $g_1, g_2 \in G$.
2. An algebra homomorphism $f : A \to B$ between two $\mathbb{C}$-linear algebras is a $\mathbb{C}$-linear map such that $f(ab) = f(a)f(b)$.
3. A monoidal functor $F : \mathcal{A} \to \mathcal{B}$ between two monoidal categories $\mathcal{A}$ and $\mathcal{B}$ is a functor equipped with isomorphisms $F(a) \otimes F(b) \simeq F(a \otimes b)$.

Question: What is the physical realization of the notion of an algebra homomorphism?

Or equivalently:

Question: How to map a quantum many-body system (or QFT) to another such that algebraic structures of operators or observables are preserved by the map?

Before we introduce this notion, we need some preparation. The term "topological order" in the following a few slides will be replaced eventually by (gapped/gapless) "quantum liquid", a notion which is, however, harder to define. Therefore, for convenience, we restrict our discussion to the familiar notion topological orders at first. We will come back to quantum liquids later.

Definition (K.-Wen:1405.5858)

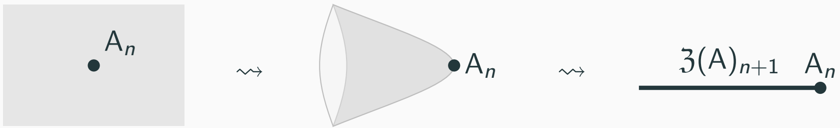
A topological order is called anomaly-free if it can be realized by a lattice model in the same dimension, and is called anomalous otherwise.

Examples: The gapped boundaries of non-trivial topological orders are examples of anomalous topological orders. A gapped boundary of the trivial n+1D topological order is an anomaly-free nD topological order.

Unique Bulk Hypothesis/Principle [K.-Wen:1405.5858]

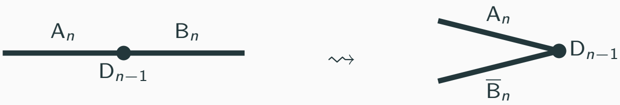
A (potentially anomalous) nD topological order A_n has a unique n+1D anomaly-free topological order as its *bulk*, denoted by $\mathfrak{Z}(\mathrm{A}_n) = \mathfrak{Z}(\mathrm{A})_{n+1}$.

Remark: An anomalous topological order must be realizable as a defect in a higher (but still finite) dimensional lattice model as illustrated below. Otherwise, it is safe to say that such a topological order does not exist. But such realizations are not unique.



After the dimensional reduction, we obtain the unique *bulk* of A_n. Importantly, A_n should be understood as a boundary phase, which includes a neighborhood of the boundary by definition.

Example: If D_{n-1} is a domain wall between A_n and B_n, then $\mathfrak{Z}(\mathrm{D})_{n+1} = \mathrm{A}_n \boxtimes \overline{\mathrm{B}}_n$.



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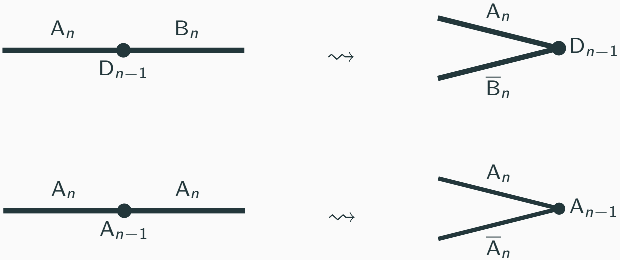
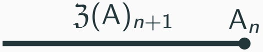


where A_{n-1} denotes the A_n restricting to the trivial domain wall of codimension 1.

Example: $\mathbb{1}_n$ = the trivial nD topological order. We have

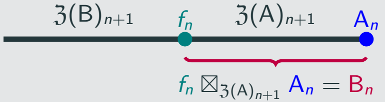
- $\mathfrak{Z}(\mathbb{1})_{n+1} = \mathbb{1}_{n+1}$;
- A_n is anomaly-free if and only if $\mathfrak{Z}(\mathrm{A})_{n+1} = \mathbb{1}_{n+1}$;
- $\mathfrak{Z}(\mathfrak{Z}(\mathrm{A}))_{n+2} = \mathbb{1}_{n+2}$.



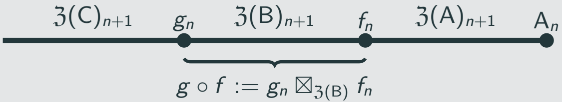
Remark: The statement of "the *bulk* of a *bulk* is trivial" is somewhat dual to that of "the boundary of a boundary is empty", which inspired homology theory. Therefore, we expect that "the *bulk* of a *bulk* is trivial" should lead us to a non-trivial "cohomology theory".

Definition (K.-Wen-Zheng:1502.01690, 1702.00673)

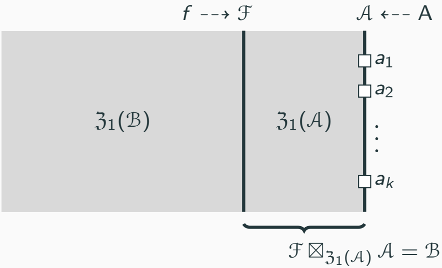
A morphism $f : \mathrm{A}_n \to \mathrm{B}_n$ between two topological orders A_n and B_n (both having gapped bulks) is a gapped wall f_n between $\mathfrak{Z}(\mathrm{A})$ and $\mathfrak{Z}(\mathrm{B})$ such that



The composition of two morphisms $f : \mathrm{A}_n \to \mathrm{B}_n$ and $g : \mathrm{B}_n \to \mathrm{C}_n$ is defined as follows:



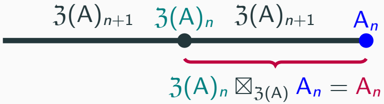
This definition of a morphism coincides with the mathematical notion of a monoidal functor between two fusion n-categories



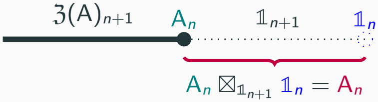
$$\begin{array}{rcl} \text{monoidal functors} & \Longleftrightarrow & \text{domain walls between } \mathfrak{Z}_1(\mathcal{B}) \text{ and } \mathfrak{Z}_1(\mathcal{A}) \\ (\mathcal{A} \xrightarrow{f} \mathcal{B}) & \longmapsto & \mathcal{F} := \mathrm{Fun}_{\mathcal{A}|\mathcal{B}}({}_f\mathcal{B}, {}_f\mathcal{B}), \\ \left(\mathcal{A} \xrightarrow{f} \mathcal{F} \boxtimes_{\mathfrak{Z}_1(\mathcal{A})} \mathcal{A} = \mathcal{B}\right) & \longleftarrow & \mathcal{F} \end{array}$$

These two constructions are inverse of each other. K.-Zheng:1507.00503, 2107.03858

Examples 1: $\mathrm{A}_n \xrightarrow{\mathrm{id_A}} \mathrm{A}_n$ is defined by the trivial domain wall $\mathfrak{Z}(\mathrm{A})_n$ in $\mathfrak{Z}(\mathrm{A})_{n+1}$.



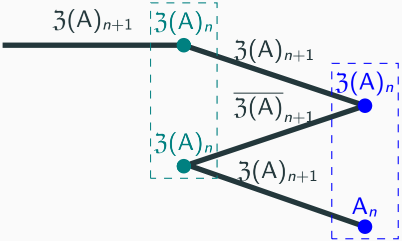
Examples 2: Let $\mathbb{1}_n$ be the trivial nD topological order. There is a canonical morphism $\iota_\mathrm{A} : \mathbb{1}_n \to \mathrm{A}_n$ defined by:



Examples 3: There is a canonical morphism

$$\mathfrak{Z}(A)_n \boxtimes A_n \xrightarrow{m} A_n$$

defined as follows



$$(\mathfrak{Z}(A)_n \boxtimes \mathfrak{Z}(A)_n) \boxtimes_{\mathfrak{Z}(A)\boxtimes\overline{\mathfrak{Z}(A)}\boxtimes\mathfrak{Z}(A)} (\mathfrak{Z}(A)_n \boxtimes A_n) = A_n$$

Theorem (K.-Wen-Zheng:1502.01690, 1702.00673)

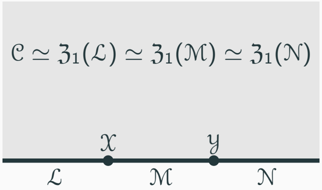
The pair $(\mathfrak{Z}(\mathrm{A})_n, m)$ *satisfies the universal property of center. That is, if* X *is an nD topological order equipped with a morphism* $f : \mathrm{X}_n \boxtimes \mathrm{A}_n \to \mathrm{A}_n$*, then there is a unique morphism* $f' : \mathrm{X}_n \to \mathfrak{Z}(\mathrm{A})_n$ *such that the following diagram is commutative:*

$$
\begin{tikzcd}
 & \mathfrak{Z}(\mathrm{A})_n \boxtimes \mathrm{A}_n \arrow[ddr, bend left, "m"] & \\
 & \mathrm{X}_n \boxtimes \mathrm{A}_n \arrow[u, "\exists!\, f' \boxtimes id_{\mathrm{A}}"] \arrow[dr, "f"] & \\
\mathrm{A}_n \arrow[uur, bend left, "\iota \boxtimes id_{\mathrm{A}}"] \arrow[ur] \arrow[rr, "id_{\mathrm{A}}"] & & \mathrm{A}_n
\end{tikzcd}
$$

Remark: This theorem simply says that the *bulk* is the center of the boundary.

This universal property works for all the well-known notions of centers.

1. When A is a group and m is a group homomorphism, it recovers the center of a group $Z(\mathrm{A}) = \{z \in \mathrm{A} | zg = gz, \forall g \in \mathrm{A}\}$.
2. When A is an algebra and m is an algebraic homomorphism, it recovers the usual center of an algebra $Z(\mathrm{A}) = \{z \in \mathrm{A} | za = az, \forall a \in \mathrm{A}\}$.
3. When A is a fusion category and m is a monoidal functor, it recovers the Drinfeld center.
4. When A is a braided fusion category and m is a braided monoidal functor, it recovers the Müger center.
5. The center of open-string CFT is the closed CFT and is called a full center. Fjelstad-Fuchs-Runkel-Schweigert:math.CT/0512076, K.-Runkel:0708.1897, Davydov:0908.1250
6. Generalized Deligne Conjecture (Kontsevich): the E_n-center of an E_n-algebra is an E_{n+1}-algebra. Lurie's book "Higher Algebras"

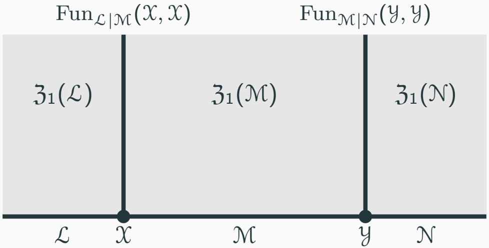


For 2+1D non-chiral topological orders, we have recovered the well-known result. In this case, a 3D topological order can be described (up to invertible ones) by a modular tensor category (MTC) $\mathcal{C}$ Moore-Seiberg:1989, Kitaev:cond-mat/0506438 and its gapped boundaries can be described by fusion categories $\mathcal{L}, \mathcal{M}, \mathcal{N}$ Kitaev-K.:1104.5047. Moreover, we have Kitaev-K.:1104.5047, K.:1307.8244

1. $\mathcal{C} \simeq \mathfrak{Z}_1(\mathcal{L}) \simeq \mathfrak{Z}_1(\mathcal{M}) \simeq \mathfrak{Z}_1(\mathcal{N})$;
2. $\mathcal{X}$ is an invertible $\mathcal{L}$-$\mathcal{M}$-bimodule that defines an Morita equivalence between $\mathcal{L}$ and $\mathcal{M}$.

Remark: Mathematically, two fusion categories are Morita equivalent if and only if they share the same center. Etingof-Nikshych-Ostrik:0809.3031.

$\mathcal{L}, \mathcal{M}, \mathcal{N}$ are fusion n-categories.
${}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}}, {}_{\mathcal{M}}\mathcal{Y}_{\mathcal{N}}$ are bimodules.



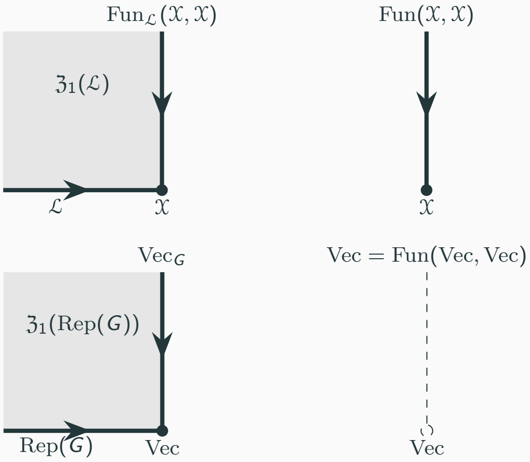
Theorem (Boundary-Bulk Relation with Defects, K.-Zheng:1507.00503, 2107.03858)

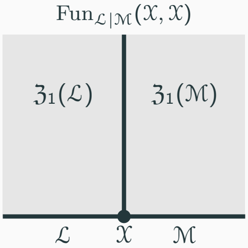
The assignment $\mathcal{L} \mapsto \mathfrak{Z}_1(\mathcal{L}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{L}}(\mathcal{L}, \mathcal{L})$ *and* $\mathcal{X} \mapsto \mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X}, \mathcal{X})$ *defines a functor.*

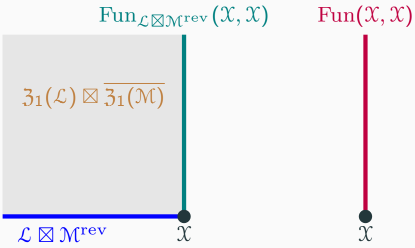
$$\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X}, \mathcal{X}) \boxtimes_{\mathfrak{Z}_1(\mathcal{M})} \mathrm{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{Y}, \mathcal{Y}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{N}}(\mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{Y}, \mathcal{X} \boxtimes_{\mathcal{M}} \mathcal{Y})$$

Fundamental formula in computing a fusion or dimensional reduction:

$$\mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X}) \boxtimes_{\mathfrak{Z}_1(\mathcal{M})} \mathrm{Fun}_{\mathcal{M}|\mathcal{N}}(\mathcal{Y},\mathcal{Y}) \simeq \mathrm{Fun}_{\mathcal{L}|\mathcal{N}}(\mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y}, \mathcal{X}\boxtimes_{\mathcal{M}}\mathcal{Y})$$

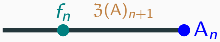






Theorem[K.-Zheng:1507.00503]: When $n=1$, this center functor is a monoidal equivalence.

Proof : $\quad {}_{\mathcal{L}}\mathcal{X}_{\mathcal{M}} = {}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}\mathcal{X} \iff$ a monodial functor : $\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}} \to \mathrm{Fun}(\mathcal{X},\mathcal{X})$

Recall : $(f : A \to B) \quad =$ 

$\iff$ a domain wall : $\mathrm{Fun}_{\mathcal{L}\boxtimes\mathcal{M}^{\mathrm{rev}}}(\mathcal{X},\mathcal{X}) = \mathrm{Fun}_{\mathcal{L}|\mathcal{M}}(\mathcal{X},\mathcal{X})$.

It automatically includes (1) $\mathcal{A} \sim^{\mathrm{Morita}} \mathcal{B}$ iff $\mathfrak{Z}_1(\mathcal{A}) \simeq \mathfrak{Z}_1(\mathcal{B})$ Etingof-Nikshych-Ostrik:0809.3031; (2) $\mathrm{BrPic}(\mathcal{A}) \simeq \mathrm{Aut}^{br}(\mathfrak{Z}_1(\mathcal{A}))$ Etingof-Nikshych-Ostrik:0909.3140. For $n>1$, it is not an equivalence.

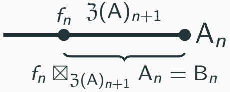
The formal proof of boundary-bulk relation assumed only the well-definedness of the notion of a morphism between two quantum phases, which is further based on the uniqueness of the bulk and the well-definedness of the fusion of domain walls. This notion and the formal proof should also work for certain 'nice' gapless phases.

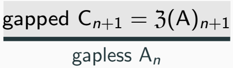
1. For 1+1D rational CFT's, this boundary-bulk relation reproduces the so-called open-closed duality for 1+1D RCFT's, i.e. the bulk (closed) CFT is the center of a boundary (or open) CFT, a result which was known long ago.
   Fjelstad-Fuchs-Runkel-Schweigert:math.CT/0512076, K.-Runkel:0708.1897, Davydov:0908.1250

Definition: Those (gapped/gapless) quantum phases such that above formal proof of boundary-bulk relation works will be called "quantum liquids", a notion which can be more precisely defined as a 'fully dualizable QFT' K.-Zheng:2011.02859

Consequences of boundary-bulk relation:

1. It leads to the classification of all n+1D anomaly-free topological orders (up to invertible ones) by fusion n-categories with a trivial E_1-center or braided fusion n-categories with a trivial E_2-center; K.-Wen:1405.5858, K.-Wen-Zheng:1502.01690, Johnson-Freyd:2003.06663
2. The proof applies to an n+1D topological order C_{n+1} with a gapless boundary A_n.





Since we already have a precise categorical description of C_{n+1} as a braided fusion n-categories $\mathcal{C}_n$ with a trivial E_2-center, then we can find the categorical description of a gapless boundary A_n by solving the mathematical equation $\mathfrak{Z}_1(?) = \mathcal{C}$.

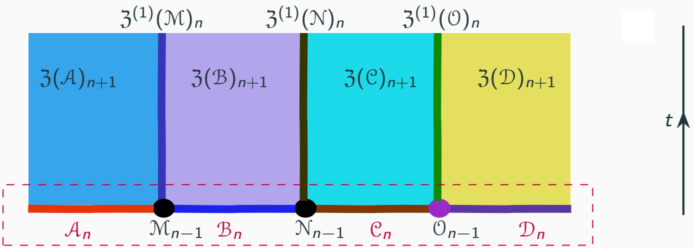
Theorem (K.-Zheng:1705.01087, 1905.04924, 1912.01760)

A gapped/gapless boundary X *of a 2+1D topological order* $(\mathcal{C}, c)$ *can be completely characterized by a pair* $\mathsf{X} = (\mathsf{X}_{\mathrm{lqs}}, \mathsf{X}_{\mathrm{top}})$, *where*

1. $\mathsf{X}_{\mathrm{lqs}} = (V, \phi)$ *is called local quantum symmetry (containing dynamical information);*
2. $\mathsf{X}_{\mathrm{top}} = {}^{\mathcal{B}}\mathcal{S}$ *is called topological skeleton.*

Moreover, $\mathsf{X}_{\mathrm{top}}$ *can be obtained by topological Wick rotation and* $\mathfrak{Z}_1(\mathsf{X}_{\mathrm{top}}) \simeq \mathcal{C}$.

When $(\mathcal{C}, c)$ *is trivial,* X *is an anomaly-free 1+1D quantum liquid (including all 1+1D rational CFT's) and* $\mathsf{X}_{\mathrm{top}} = {}^{\mathfrak{Z}_1(\mathcal{S})}\mathcal{S}$.



We denote the category of nD anomaly-free quantum liquids by QL^n (morphisms are domain walls) and that of the topological skeletons of nD quantum liquids by $\mathrm{QL}^n_{\mathrm{top}}$.

Theorem (K.-Zheng:2011.02859, 2201.05726)

$$\mathrm{QL}^n \simeq \mathrm{QL}^n_{\mathrm{top}} \simeq \bullet/(n+1)\mathrm{Vec},$$

where $(n+1)\mathrm{Vec} = \Sigma^n\mathrm{Vec} = \Sigma^{n+1}\mathbb{C}$ *is the category of nD non-chiral topological orders (that admit gapped boundaries) and higher codimensional defects.* *Gaiotto-Johnson-Freyd:1905.09566.*

Summary: Main goal of this talk is to introduce the notion of a morphism between QFT's, and to show that it is useful and powerful. Indeed, it alone had led us to the formal proof of boundary-bulk relation, to the discovery of topological Wick rotation, and to a unified mathematical theory of gapped and gapless quantum liquids, and to the study of the categories of quantum liquids.

Similar to the fact that category theory once revolutionized algebraic geometry by Grothendieck and his school, we believe that it will also revolutionize the theories of QFTs, phase transitions and perhaps quantum gravity. The important thing is not only to borrow categorical language for physical use, but also to use the spirit of the category theory to ask new questions and find new truths. To define the notion of a morphism between QFT's is only an example of many possibilities.

If the notion of a morphism is arguably the most important concept in mathematics, it is only reasonable that the notion of a morphism between QFT's is an important concept in physics.

Thank you !