

Summary of Liang Kong's Research

Liang Kong *

International Quantum Academy, Shenzhen 518048, China

After some years in studying condensed matter physics, for lack of new ideas in solving strong coupling problems (e.g., high T_c superconductors) and phase transitions, I decided, in 1996, to look for new ideas from other fields, such as high energy physics and mathematical physics. In the fall of 1997, I transferred to Rutgers to study string theory at the height of the second string revolution. Although I was excited by the new developments, such as mirror symmetry, D-branes, Seiberg-Witten theory, M-theory, AdS/CFT correspondence, microscopic origin of black hole entropy, non-commutative geometry, I was also overwhelmed and unable to tell which topics is important or necessary. After some intensive studies of various frontier topics and some careful evaluations, partly influenced by my old interests in phase transitions and an old lesson that simple systems nourish new methodologies, I decided to step back and restudy 2D rational conformal field theories (RCFT), a subject which was the base for many new developments but was regarded by most working physicists as something understood and outdated. Since RCFT's were actively pursued only by mathematicians in 90's, I decided to pursue a Ph.D degree in mathematics. This is the beginning of a long journey of simplification from 2D RCFT's to TQFT's then to topological orders, which are the simplest quantum phases or QFT's (as TQFT's defined on an open disk). This simplification leads to revolutionary new insights, languages and methodologies in the study of quantum many-body systems or QFT's, including a unified mathematical theory of gapped/gapless quantum liquids¹ in all dimensions. We use nD to represent the spacetime dimension.

- (1) **Mathematical Foundation of Open-closed Rational Conformal Field Theories:** Together with my Ph.D advisor Yi-Zhi Huang, we started the program to establish the mathematical foundation of open-closed RCFT based on the mathematical theory of vertex operator algebra (VOA) and its representation theory². One motivation of this program is to understand the mathematical meaning of a D-brane and open-closed duality. This program was successfully accomplished in a series of papers [HK04, HK07, Kon07, HK10, Kon08a, Kon08b] (see [Kon11] for a review).
 - (a) In [HK04], we introduced the notion of an open-string vertex operator algebra V_{op} , which is the mathematical definition of an open (or boundary) CFT. Under some natural assumptions, we proved that V_{op} is an algebra over the so-called disk partial operad. We also give V_{op} a tensor-categorical description, as an algebra in the category Mod_V of V -modules when V_{op} contains a rational VOA V in its meromorphic center and Mod_V is a modular tensor category.
 - (b) In [HK07], we introduced the notion of a full field algebra, which is a non-chiral analogue of a VOA, and gave some rigorous constructions of full field algebras. In [Kon07], we proved that a full field algebra over $V \otimes_{\mathbb{C}} \bar{V}$, where V is a rational VOA, is equivalent to a commutative algebra in $\text{Mod}_V \boxtimes \text{Mod}_{\bar{V}}$, and gave some categorical construction of such full field algebras.

*Email: kongustc@outlook.com

¹A gapped/gapless quantum liquid can be viewed as a fully dualizable gapped/gapless QFT.

²There are another two successful approaches towards 2D RCFT's: (1) Conformal-net approach, in which a CFT is defined by a conformal net; (2) Fuchs-Runkel-Schweigert's approach, in which a 2D RCFT is formulated as a natural transformation from the trivial functor to a non-trivial one obtained by restricting the Reshetikhin-Turaev TQFT functor to the 1-category of surfaces (as objects) and their self-sewings as 1-morphisms [FFRS08].

In [HK10], we prove that full field algebras constructed in [HK07] are modular invariant, thus are the candidates of modular-invariant closed (or bulk) CFT's.

- (c) In [Kon08a], by combining the open parts with the closed parts, we introduced the notion of an open-closed field algebra, and proved that it gives an algebra over Swiss-Cheese partial operad. We also gave a tensor-categorical formulation and construction of such an algebra when it satisfies a V -symmetric boundary condition for a rational VOA V .
 - (d) In [Kon08b], we gave a precise mathematical formulation of the modular-invariant condition and the Cardy condition for an open-closed field algebra. When it satisfies a V -symmetric boundary condition for a rational VOA V , we obtain rigorously a categorical reformulation of these conditions. Up to certain convergence and modular-invariance assumptions in higher genus surfaces, we obtained a complete tensor-categorical reformulation of an open-closed rational CFT in terms of generators and relations. It was slightly refined in [KR09]. This result automatically includes Moore-Segal's theorem on open-closed TFT's in terms of generators and relations as a special case. We also gave rigorous constructions.
 - (e) In [KLR14], we proved rigorously that the tensor-categorical reformulation of an open-closed RCFT obtained in our approach in terms of generators and relations [Kon08b, KR09] is equivalent to Fuch-Runkel-Schweigert's definition of an open-closed RCFT.
 - (f) In [Kon11], I reviewed the main results in above series of works and results on open-closed duality [KR08, KR09], and proposed a new foundation of geometry (called *stringy algebraic geometry*) based on 2D conformal field theories and D-branes.
- (2) **Mathematical Theory of Open-closed Duality:** In a series of papers [KR08, KR09, DKR11a, DKR11b, DKR15, KYZ21], joint with a few different collaborators, we formulated precisely, proved rigorously and further developed the open-closed duality (or boundary-bulk relation) in open-closed rational CFT's. This development went through three different phases.
- (a) In the first phase, the open-closed duality, i.e., bulk is the center of a boundary, is limited to a single bulk CFT without topological defect lines. In [KR08], we proved, within the framework of RCFT's, that two open CFT's are Morita equivalence if³ and only if their full centers (or equivalently, their bulk CFT's) are isomorphic. In [KR09], we show that a simple modular-invariant closed (or bulk) CFT is necessarily equipped with consistent open (or boundary) CFT's. Moreover, we also proved that a modular-invariant closed CFT is precisely a Lagrangian algebra⁴ in $\text{Mod}_V \boxtimes \overline{\text{Mod}}_V$.
 - (b) In the second phase, hidden structure as a higher layer of open-closed duality was first observed in [DKR11a]; then we introduced and proved in [DKR15] the functoriality of the open-closed duality by including topological defect lines in a single bulk CFT. In [DKR11b], we applied this functoriality to 2D TQFT's by including topological defects lines between different bulk 2D TQFT's evaluated in a 2-category⁵.
 - (c) In the third phase, based on some new developments in the boundary-bulk relation in the field of 2+1D topological orders [KWZ15, KZ18a], we introduced and proved in [KYZ21] the functoriality of the open-closed duality by including domain walls between different bulk CFT's.

- (3) **Boundary-bulk relation in Topological orders:** The reason I have been working on open-closed duality for so long is because it plays the role of equivalence principle in my program of quantum gravity or stringy algebraic geometry. This program was envisioned in 2007 and was invited to be

³The 'if' part was obtained earlier by Frölich, Fuchs, Runkel and Schweigert in [FFRS07].

⁴The name of 'Lagrangian algebra' was coined later in [DMNO13].

⁵The axiomatic study of 2D TQFT with topological defect lines in this work [DKR11b] was partly rediscovered in 2017-2019 under a new name: a 2D TQFT with a fusion category symmetry [BT17, TW19].

published in [Kon11]. In the spring of 2008, I met Alexei Kitaev. He told me a mathematical result: two fusion categories are Morita equivalent if and only if they share the same Drinfeld center⁶ and its physical meaning. It was well-known that anyons a 2+1D topological order form a modular fusion category \mathcal{C} . When this topological order admits a gapped boundary, \mathcal{C} is precisely the Drinfeld center $\mathfrak{Z}_1(\mathcal{A})$ of the fusion 1-category \mathcal{A} consisting of particles on a gapped boundary, i.e., $\mathcal{C} \simeq \mathfrak{Z}_1(\mathcal{A})$. This immediately suggests that “bulk = \mathfrak{Z} (boundary)” might be a universal principle holds for many quantum field theories. This vision led to a series of works on the boundary-bulk relation for topological orders in all dimensions [KK12, Kon14a, KWZ15, KZ18a, KZ24].

- (a) Kitaev’s arguments for the physical interpretation is formal and model independent. In order to understand it better, I decided to check it via concrete lattice models. This led to a joint work [KK12], in which we explicitly proved “bulk = \mathfrak{Z} (boundary)” in generalized Levin-Wen models including gapped boundaries. Moreover, [KK12] consists of lattice model constructions of all gapped domain walls and higher codimensional gapped defects in any 2+1D non-chiral topological order. These defects form a monoidal 2-category, which turns out to be the first examples of fusion 2-categories [DR18].
 - (b) In [KWZ15], we provided a proof of the boundary-bulk relation (i.e., “bulk = \mathfrak{Z} (boundary)”) for topological orders in all dimensions based on the key observation that the notion of a center is determined by its universal property expressed in terms of morphisms between two mathematical objects of the same type. Therefore, it is enough to introduce a physical definition of a morphism between two (potentially anomalous) topological orders then check the universal property of center directly. Indeed, in [KWZ15], we introduced the notion of a morphism between two topological orders, which is obviously generalizable to other quantum field theories (see [KWZ17, KZ22a]), and checked that the universal property of center holds tautologically. We proved later that this notion of a morphism coincides with the notion of a monoidal functor between two fusion n -categories [KZ18a, KZ24]. Moreover, we conjectured the functoriality of boundary-bulk relation in [KWZ15].
 - (c) Similar to the functoriality of the open-closed duality in RCFT’s or TQFT’s [DKR15, DKR11b], in [KZ18a], we rigorously proved the functoriality of the boundary-bulk relation for 2+1D topological orders by showing that the Drinfeld center can be extended to a center functor, which can be formulated and proved to be a symmetric monoidal equivalent. Moreover, this functoriality provides a fusion formula that is a powerful tool to compute factorization homology on surfaces [AKZ17]. This functoriality was later generalized to higher dimensions in [KZ24].
- (4) **A Unified Mathematical Theory of All Gapped Quantum Liquids:** A gapped quantum liquid is a notion that summarizes all topological orders, SPT/SET orders and spontaneous symmetry breaking (SSB) orders [ZW15, KLVZZ20a]. We have successfully found a unified mathematical characterization of a gapped quantum liquid with a finite internal symmetry in any dimensions based on three guiding principles: remote detectable principle [KW14], boundary-bulk relation [KWZ15, KWZ17] and condensation completion principle [KTZ20, GJF19, JF22, KLVZZ20a].
- (a) The first mathematical characterization of 2+1D bosonic SPT/SET orders was obtained in [BBCW19] in terms of G -crossed braided fusion categories. We developed a unified mathematical characterization of 2+1D bosonic/fermionic SPT/SET orders based on minimal modular extensions [LKW17a, LKW17b] such that it is equivalent to results in [BBCW19] for bosonic SPT/SET orders. Mathematically, we proved that minimal modular extensions of a symmetry fusion category \mathcal{E} form a finite abelian group $\text{Mext}(\mathcal{E})$; and for a braided fusion category with Müger center given by \mathcal{E} , the set of its minimal modular extensions is either empty or a $\text{Mext}(\mathcal{E})$ -torsor.

⁶This results was independently obtained by Etingof, Nikshych and Ostrik in the summer of 2008 in [ENO10].

- (b) In [KW14, KWZ15], we initiated a program of studying topological orders in all dimensions as a whole, or more precisely, studying the categories of all topological orders. Many questions and conjectures are raised in these works. In particular, the mathematical characterization of a topological order (up to invertible ones) were correctly conjectured in [KWZ15] without knowing how to define a multi-fusion n -category. The most surprising and influential result along this line was the complete classification of all 3+1D TQFT's obtained in [LKW18, LW19]. In particular, the main result of [LKW18] says that a 3+1D topological orders with only bosonic particles is necessarily a twist finite gauge theory, whose modular 2-category of topological defects of codimension 2 and higher was rigorously computed later in [KTZ20]. This result was proved at a physical level of rigor in [LKW18] and was later rigorously proved by Johnson-Freyd in [JF20].
- (c) In spite of above success, [KW14, KWZ15] miss some fundamental ingredients. One of the missing ingredients is the mathematical theory of condensation completion first developed for 2-categories by Carqueville and Runkel in [CR16] and further developed by Douglas and Reutter in [DR18] then developed for n -categories by Gaiotto and Johnson-Freyd in [GJF19]. This theory leads to a mathematical theory of multi-fusion n -category developed in [DR18] for $n = 2$ and in [JF22] for all n . A rather complete mathematical characterization of topological orders (up to invertible ones) was fully established by Johnson-Freyd in [JF22]. Main conjectures in [KWZ15] were proved in [JF22]. Another paper [KLWZZ20a], which appeared in the same week as [JF22], developed the mathematical characterization of SPT/SET orders. It contains that of topological orders as a special case, which was largely taken from granted in [KLWZZ20a]. Johnson-Freyd's theory of multi-fusion n -category was later simplified in [KZ22a] via the theory of separable n -categories, which is also based on [GJF19].
- (d) In [KLWZZ20a, KLWZZ20b], we found a unified mathematical characterization of a gapped quantum liquid (including topological/SPT/SET/SSB orders) with a finite internal symmetry. Among many other things, this theory implies a topological holography between an $n+1$ D topological order together with a fixed gapped boundary and an n D SPT/SET/SSB orders.
- (5) **Gapless Boundaries of 2+1D Topological Orders:** A surprising observation in [KWZ17] is that the proof of “bulk = \mathfrak{Z} (boundary)” [KWZ15, KWZ17] applies to gapless boundaries of topological orders. This immediately suggests that there is a unified mathematical theory of gapped and gapless boundaries of (potentially trivial) topological orders in all dimensions.
- (a) In a chiral 2+1D topological order, we know its anyons form a modular fusion 1-category \mathcal{C} and its boundaries are topologically protected gapless. By “bulk = \mathfrak{Z} (boundary)”, something on the gapless boundary should form a categorical structure such that its monoidal center is \mathcal{C} . In [KZ18c], we found this categorical structure (i.e., a fusion 1-category enriched in a modular fusion 1-category) through a careful analysis of all possible macroscopic observables on the 1+1D world sheet of the boundary. As a consequence, we established for the first time a unified mathematical theory of gapped and gapless boundaries of 2+1D topological orders. When the bulk topological order is trivial, this theory automatically recovers all the previous results on anomaly-free 1+1D RCFT's. As a byproduct, we obtain a mysterious holography called “Topological Wick Rotation”, which is a holography between (3D topological order + a gapped boundary) and the topological data of a 2D RCFT's.
- (b) In [KZ20, KZ21], we provide a complete ‘physical proof’ of this theory by providing all necessary mathematical details and minimizing non-mathematical arguments to either a no-go theorem or a physical principle. Moreover, we proposed in [KZ20, KZ21] that the topological Wick rotation naturally generalizes to all dimensions with 2D RCFT's replaced by gapless quantum liquids.

These works provide a solid evidence that there are a single mathematical theory for topological orders and their phase transitions (which close the energy gap). In [CJKYZ20], we gave an illustrating example of a purely boundary phase transition between the smooth boundary and the rough boundary of the 2+1D toric code model, where we have a complete mathematical characterization of the critical point regarded as a special case of gapless boundaries of toric code model.

- (6) **A New Theory of Enriched Categories:** As we mention before, the notion of a fusion category enriched in a braided fusion category naturally arises in the study of gapless boundaries of 2+1D topological orders [KZ18c]. This notion was independently introduced by mathematicians only a few months earlier [MP19]. However, physics demands a far deeper and richer theory than mathematicians' expectation. Among many other things, physics demands a new theory of enriched categories, in which we should allow the ambient categories to change freely. Two of our works on enriched categories illustrate and emphasize this fact with deep physical applications.
- (a) In [KZ18c], we introduced a constructive definition of the monoidal center of a monoidal category enriched in a braided monoidal category, and proved some general theorems for computing such center. Among many other things, this paper showed that the monoidal center is often enriched in a different braided monoidal category. For example, for a modular fusion category \mathcal{C} , by replacing its hom space by internal homs, we obtain an \mathcal{C} -enriched fusion category ${}^{\mathcal{C}}\mathcal{C}$, where $\text{hom}_{e_{\mathcal{C}}}(a, b) = [a, b] = b \otimes a^*$ for $a, b \in \mathcal{C}$. In this case, we have $\mathfrak{Z}_1({}^{\mathcal{C}}\mathcal{C}) \simeq \mathcal{C}$, where \mathcal{C} is enriched in the category Vec of finite dimensional vector spaces over \mathbb{C} . This fact has a deep physical meaning and also a mathematical meaning because it suggests how to extend a Reshetikhin-Turaev TQFT down to a point, i.e., by assigning a point to ${}^{\mathcal{C}}\mathcal{C}$ [Zhe17].
 - (b) Although above work [KZ18c] is enough to demonstrate our point, it does not reveal the deeper reason for such interesting phenomena to occur. In [KZZZ24], we reveal that there is a natural 2-category of enriched categories, where the notion of a functor is defined for two enriched categories with different ambient categories. This notion largely generalizes the usual notion of a functor used in the literature or textbook on enriched categories. This work shows that the theory of enriched categories should be developed within this 2-category. This work [KZZZ24] is only the first in a series, where we develop a new theory of (braided monoidal) enriched categories. The second in this series is on the representation theory of enriched monoidal categories and was partly predicted in our physics paper [KZ21].
 - (c) In [KZ20, KZ21], we propose that topological Wick rotations naturally generalize to all dimensions. This demands a yet-unknown mathematical theory of enriched higher categories, a notion which still does not have a mathematical definition. This theory should be an indispensable ingredient of a complete new paradigm of phases and phase transitions.
- (7) **A Unified Mathematical Theory of Gapped/Gapless Quantum Liquids:** Based on the mathematical theory of condensation completion [CR16, DR18, GJF19, JF22], by combining [KZ18c, KZ18b, KZ20, KZ21] with [KLWZZ20a, KLWZZ20b], we developed a unified mathematical theory of all gapped/gapless quantum liquids in a trilogy [KZ22a, KZ24, KZ22b].
- (a) In the first of the trilogy [KZ22a], we refine Johnson-Freyd's theory on multi-fusion n -categories [JF22] by developing the theory from a slightly different perspective. More precisely, we first developed a theory of separable n -categories, then build that of E_k -fusion n -categories on the top of it. We recover many results in [JF22] and prove many new results on E_k -fusion n -categories and E_k -centers. Moreover, we use it to build a unified mathematical theory of all quantum liquids. In particular, an nD quantum liquid X^n can be completely characterized by a pair $(X_{\text{lqs}}, X_{\text{top}})$, where X_{lqs} summarizes a minimal dynamical data and is called local

quantum symmetry and X_{top} summarizes all the topological data and is called the topological skeleton of X^n . We compute explicitly the categories of topological skeletons of all nD quantum liquids, and show that these categories can be obtained from the one for $n = 1$ by iterated delooping. This work also proposed for the first time that all macroscopic observables in a 2D SPT/SET/SSB order form a fusion category enriched in a modular fusion category. This proposal was later validated in Ising chain and Kitaev chain [KWZ22] and in other 2D lattice models in [XZ24].

- (b) In the second of the trilogy [KZ24], we proved the functoriality of the E_0 -centers of separable n -categories and derived that of the E_1 -centers of fusion $(n-1)$ -categories as a consequence. This new approach dramatically simplified the proof of the functoriality of the Drinfeld centers of fusion 1-categories obtained in [KZ18c]. We further study E_k -fusion n -categories and develop a theory of E_k -centralizers, and use it to prove some fundamental theorems on minimal modular extensions. We also establish some basic facts on group-theoretical higher categories. In the end, we develop a mathematical theory of SPT/SET order with a finite spacetime symmetry. In particular, we predicts some new SPT orders.
- (c) In the third of the trilogy [KZ24], we developed the mathematical theory of local quantum symmetries by generalizing the theory of conformal nets in 2D to that of topological nets in higher dimensions. As a consequence, we obtained a rather complete theory of all gapped/gapless quantum liquids. It allows us study dualities, holography, dimension reductions, symmetry-breaking/condensation/gauging, boundary-bulk duality and anomalies all within a single theoretical framework. It is rather surprising that such a theory is even possible. We are still in a beginning stage to understand what is really going on.

It is important to remark that such a powerful framework is still not enough to cover a complete mathematical theory of phase transitions, which demands us to go beyond quantum liquids to include certain non-liquid phases. This will be the subject to study in the next a few years.

(8) **A Unified Mathematical Theory of Defect Condensations:** A complete mathematical theory of phase transitions among gapped quantum liquids demands a new calculus, in which we can barely understand integers. It means that there are a lot of new mathematics needed in order to achieve this goal. It was known that many phase transitions among topological orders can driven by the condensation of topological defects. The bootstrap study of defect condensations can be accomplished within the framework of quantum liquids.

- (a) The first example of defect condensation was known as the so-called ‘anyon condensation’ in 2+1D topological order. Originated from Moore and Seiberg’s work on the extension of chiral algebra in RCFT’s [MS89b, MS89c], the physical theory of anyon condensation was officially introduced by Bais, Schroers, Slingerland in 2002 [BSS02, BSS03]. After some success in the special case of finite abelian gauge theories [KS11, Lev13, BJQ13], a complete mathematical theory of anyon condensations in all 3D topological orders was established in my work [Kon14b], which was based on many earlier mathematical works [KO02].
- (b) In [KZZZ24], we develop a unified mathematical theory of defect condensations for topological orders in all dimensions based on higher categories, higher algebras and higher representations. A k -codimensional topological defect A in an $n+1D$ (potentially anomalous) topological order C^{n+1} is condensable if it is equipped with the structure of a condensable E_k -algebra. Condensing such a defect A amounts to a k -step process. In the first step, we condense the defect A along one of its transversal directions, thus obtaining a $(k-1)$ -codimensional defect ΣA , which is naturally equipped with the structure of a condensable E_{k-1} -algebra. In the second step, we condense the defect ΣA in one of the remaining transversal directions, thus obtaining a $(k-2)$ -codimensional defect $\Sigma^2 A$, so on and so forth. In the k -th step, we

condense the 1-codimensional defect $\Sigma^{k-1}A$ along the only transversal direction, thus defining a phase transition from C^{n+1} to a new $n+1$ D topological order D^{n+1} . We give precise mathematical descriptions of each step in above process, including the precise mathematical characterization of the condensed phase D^{n+1} . We also briefly predicted some results in higher Morita theory, a theory of integrals (generalizing that of factorization homology) and a theory of the condensations of gapless liquid-like defects.

The bootstrap results of defect condensations tell us the relation between the initial phase and the condensed phase. But this theory is not enough to determine the critical point. A more complete theory of phase transitions will be our focus of study in the next a few years.

Remark 0.0.1. Since 2022, some results obtained in (3), (4) and (5) have been gradually rediscovered under the name “sandwich construction”, “topological symmetry” [FMT22] or “SymTFT” [ABGE+23], and have become one of the hottest topics in theoretical physics. After over 100 research papers published in this direction since 2022, as far as I can tell, some important results in (4), (5) and (7) are still largely unknown to physics community. It is worth pointing out that these results (e.g., “SymTFT” and “topological holography”) appeared in this new wave of developments after 2022 as new proposals or new ideas but appeared in our works as the consequences of a much more complete theory. \diamond

I have summarized all my previous works. Although the unified mathematical theory of all quantum liquids (including SPT/SET/SSB order and gapless liquids) and that of defect condensations are breakthroughs that are hard to imagine ten years ago, they are only the beginning of an ongoing journey towards to a revolutionary new calculus of quantum many-body theory or QFT’s [Kon24]. I believe that quantum gravity is impossible without this new calculus [Kon20]. My long term goal is to establish this new calculus.

References

- [AKZ17] Yin-Hua Ai, Liang Kong, Hao Zheng, *Topological orders and factorization homology*, Adv. Theor. Math. Phys. Vol. 21, Number 8, (2017), 1845-1894, [arXiv:1607.08422](https://arxiv.org/abs/1607.08422)
- [ABGE+23] F. Apruzzi, F. Bonetti, I. García Etxebarria, S. S. Hosseini, and S. Schafer-Nameki, *Symmetry TFTs from String Theory*, Commun. Math. Phys. 402, 895–949 (2023). <https://doi.org/10.1007/s00220-023-04737-2>
arXiv:2112.02092 [hep-th].
- [BSS02] F. A. Bais, B. J. Schroers, and J. K. Slingerland. Broken quantum symmetry and confinement phases in planar physics. Physical Review Letters, 89(18), 2002. [arXiv:hep-th/0205117](https://arxiv.org/abs/hep-th/0205117).
- [BSS03] Alexander F Bais, Bernd J Schroers, and Joost K Slingerland. *Hopf symmetry breaking and confinement in (2+1)-dimensional gauge theory*. Journal of High Energy Physics, 2003(05):068–068, 2003. [arXiv:hep-th/0205114](https://arxiv.org/abs/hep-th/0205114)
- [BBCW19] Maissam Barkeshli, Parsa Bonderson, Meng Cheng, and Zhenghan Wang. *Symmetry fractionalization, defects, and gauging of topological phases*. Physical Review B, 100(11), 2019. [arXiv:1410.4540](https://arxiv.org/abs/1410.4540)
- [BJQ13] Maissam Barkeshli, Chao-Ming Jian, and Xiao-Liang Qi. *Theory of defects in abelian topological states*. Physical Review B, 88(23), 2013. [arXiv:1305.7203](https://arxiv.org/abs/1305.7203)

- [BCKA13] Oliver Buerschaper, Matthias Christandl, Liang Kong, Miguel Aguado, *Electric-magnetic duality and topological order on the lattice*, Nuclear Physics B. 876 [FS] (2013) 619-636 [arXiv:1006.5823](#)
- [BT17] Lakshya Bhardwaj, Yuji Tachikawa, *On finite symmetries and their gauging in two dimensions*, J. High Energ. Phys. 2018, 189 (2018). [https://doi.org/10.1007/JHEP03\(2018\)189](https://doi.org/10.1007/JHEP03(2018)189), [arXiv:1704.02330](#)
- [CJKYZ20] W.-Q. Chen, C.-M. Jian, L. Kong, Y.-Z. You, Hao Zheng, *A topological phase transition on the edge of the 2d \mathbb{Z}_2 topological order*, Phys. Rev. B 102, 045139 (2020) [arXiv:1903.12334](#)
- [CR16] Nils Carqueville and Ingo Runkel. *Orbifold completion of defect bicategories*. Quantum Topology, 7(2):203–279, 2016. [arXiv:1210.6363](#)
- [DKR11a] Alexei Davydov, Liang Kong, Ingo Runkel, *Invertible defects and isomorphisms of rational CFTs*, Adv. Theor. Math. Phys., 15, (2011) 43–69 [arXiv:1004.4725](#)
- [DKR11b] Alexei Davydov, Liang Kong, Ingo Runkel, *Field theories with defects and the centre functor*, Mathematical Foundations of Quantum Field and Perturbative String Theory, Hisham Sati, Urs Schreiber (eds.), Proceedings of Symposia in Pure Mathematics, AMS Vol. 83 (2011) 71–128 [arXiv:1107.0495](#)
- [DKR15] Alexei Davydov, Liang Kong, Ingo Runkel, *The functoriality of the centre of an algebra*, Adv. Math. 285 (2015) 811-876, [arXiv:1307.5956](#)
- [DMNO13] Alexei Davydov, Michael Müger, Dmitri Nikshych, and Victor Ostrik, *The Witt group of non-degenerate braided fusion categories*, Journal für die reine und angewandte Mathematik (Crelles Journal), 2013(677), 2013. [arXiv:1009.2117](#)
- [DR18] Christopher L. Douglas and David J. Reutter. *Fusion 2-categories and a state-sum invariant for 4-manifolds*. [arXiv:1812.11933](#)
- [ENO10] P. Etingof, D. Nikshych, V. Ostrik, *Weakly group-theoretical and solvable fusion categories*, Adv. Math. 226(1), 176–205 (2010) [arXiv:0809.3031](#)
- [FFRS07] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, *Duality and defects in rational conformal field theory*. Nucl. Phys. B 763, 354–430 (2007) [arXiv:hep-th/0607247](#)
- [FFRS08] J. Fjelstad, J. Fuchs, I. Runkel, C. Schweigert, *Uniqueness of open/closed rational CFT with given algebra of open states*, Adv. Theor. Math. Phys. 12 (2008) 1283–1375, [arXiv:hep-th/0612306](#).
- [FMT22] Daniel S. Freed, Gregory W. Moore, and Constantin Teleman. *Topological symmetry in quantum field theory*. [arXiv:2209.07471](#)
- [GJF19] Davide Gaiotto, Theo Johnson-Freyd, *Condensations in higher categories*, [arXiv:1905.09566](#)
- [HK04] Yi-Zhi Huang, Liang Kong *Open-string vertex algebras, categories and operads*, Commun. Math. Phys. 250 (2004) 433-471, [arXiv:math/0308248](#)
- [HK07] Yi-Zhi Huang, Liang Kong, *Full field algebras*, Commun. Math. Phys. 272 (2007) 345–396, [arXiv:math/0511328](#)
- [HK10] Yi-Zhi Huang, Liang Kong, *Modular invariance for conformal full field algebras*, Trans. Amer. Math. Soc. 362 (2010) 3027–3067, [arXiv:math.QA/0609570](#)
- [JF22] Theo Johnson-Freyd, *On the classification of topological orders*, Commun. Math. Phys., 2022, [arXiv:2003.06663](#)

- [JF20] Theo Johnson-Freyd. *(3+1)D topological orders with only a \mathbb{Z}_2 -charged particle*. [arXiv:2011.11165](#)
- [KS11] Anton Kapustin and Natalia Saulina. *Topological boundary conditions in abelian Chern-Simons theory*. Nuclear Physics B, 845(3):393–435, 2011. [arXiv:1008.0654](#)
- [KO02] A. Kirillov Jr., V. Ostrik, *On a q -Analogue of the McKay Correspondence and the ADE Classification of $sl(2)$ Conformal Field Theories*, Adv. Math. Vol. 171, Issue 2, (2002) 183-227 [arXiv:math/0101219](#)
- [KK12] Alexei Kitaev, Liang Kong, *Models for gapped boundaries and domain walls*, Commun. Math. Phys. 313 (2012) 351-373 [arXiv:1104.5047](#)
- [Kon07] Liang Kong, *Full field algebras, operads and tensor categories*, Adv. Math. 213 (2007) 271–340, [arXiv:math/0603065](#)
- [Kon08a] Liang Kong, *Open-closed field algebras*, Commun. Math. Phys., 280, 207-261 (2008) [arXiv:math.QA/0610293](#)
- [Kon08b] Liang Kong, *Cardy condition for open-closed field algebras*, Commun. Math. Phys., 283, 25–92 (2008) [arXiv:math/0612255](#)
- [Kon11] Liang Kong, *Conformal field theory and a new geometry*, Mathematical Foundations of Quantum Field and Perturbative String Theory, Hisham Sati, Urs Schreiber (eds.), Proceedings of Symposia in Pure Mathematics, AMS, Vol. 83 (2011) 199–244 [arXiv:1107.3649](#)
- [Kon14a] Liang Kong, *Some universal properties of Levin-Wen models*, Proceedings of XVIITH International Congress of Mathematical Physics, World Scientific 444-455 (2014) [arXiv:1211.4644](#)
- [Kon14b] Liang Kong, *Anyon condensation and tensor categories*, Nucl. Phys. B 886 (2014) 436-482, [arXiv:1307.8244](#)
- [Kon20] Liang Kong, *Infinitely more is more different — why shall we care about topological orders?*, Online Talk in 2020, <https://www.koushare.com/live/details/42?vid=5321>
- [Kon24] Liang Kong, *An ongoing revolution in quantum many-body theory*, an online talk (in Chinese), Oct. 15, 2024.
- [KLWZZ20a] Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng, *Classification of topological phases with finite internal symmetries in all dimensions*, J. High Energ. Phys., 2020, 93 (2020) [arXiv:2003.08898](#)
- [KLWZZ20b] Liang Kong, Tian Lan, Xiao-Gang Wen, Zhi-Hao Zhang, Hao Zheng, *Algebraic higher symmetry and categorical symmetry: a holographic and entanglement view of symmetry*, Phys. Rev. Research 2, 043086 (2020) [arXiv:2005.14178](#)
- [KLR14] Liang Kong, Qin Li, Ingo Runkel, *Cardy algebras and sewing constraints, II*, Adv. Math. 262 (2014) 604-681 [arXiv:1310.1875](#)
- [KR08] Liang Kong, Ingo Runkel, *Morita classes of algebras in modular tensor categories*, Adv. Math. 219, 1548–1576 (2008) [arXiv:0708.1897](#)
- [KR09] Liang Kong, Ingo Runkel, *Cardy algebras and sewing constraints, I*, Commun. Math. Phys. 292, 871–912 (2009) [[arXiv:0807.3356](#)]
- [KTZ20] Liang Kong, Yin Tian, Shan Zhou, *The center of monoidal bicategories in 3+1D Dijkgraaf-Witten Theory*, Adv. Math 360 (2020) 106928 [arXiv:1905.04644](#)

- [KTZha20] Liang Kong, Yin Tian, Zhi-Hao Zhang, *Defects in the 3-dimensional toric code model form a braided fusion 2-category*, J. High Energ. Phys. 2020, 78 (2020). [arXiv:2009.06564]
- [KW14] Liang Kong, Xiao-Gang Wen *Braided fusion categories, gravitational anomalies and the mathematical framework for topological orders in any dimensions*, arXiv:1405.5858
- [KWZ15] Liang Kong, Xiao-Gang Wen, Hao Zheng, *Boundary-bulk relation for topological orders as the functor mapping higher categories to their centers*, arXiv:1502.01690
- [KWZ17] Liang Kong, Xiao-Gang Wen, Hao Zheng, *Boundary-bulk relation in topological orders*, Nucl. Phys. B 922 (2017), 62-76, arXiv:1702.00673
- [KWZ22] Liang Kong, Xiao-Gang Wen, Hao Zheng, *One dimensional gapped quantum phases and enriched fusion categories*, J. High Energ. Phys., 2022, 22 (2022) arXiv:2108.08835
- [KYZZ24] Liang Kong, Wei Yuan, Zhi-Hao Zhang, Hao Zheng, *Enriched monoidal categories I: centers*, Quantum Topol. (2024) DOI10.4171/QT/217, arXiv:2104.03121
- [KYZ21] Liang Kong, Wei Yuan, Hao Zheng, *Pointed Drinfeld center functor*, Commun. Math. Phys. published online: <https://doi.org/10.1007/s00220-020-03922-x> (2021) arXiv:1912.13168
- [KZha22] Liang Kong, Zhi-Hao Zhang, *An invitation to topological orders and category theory*, arXiv:2205.05565
- [KZZZ24] Liang Kong, Zhi-Hao Zhang, Jiaheng Zhao, Hao Zheng, *Higher condensation theory*, arXiv:2403.07813
- [KZ17] Liang Kong, Hao Zheng, *Semisimple and separable algebras in multi-fusion categories*, arXiv:1706.06904
- [KZ18a] Liang Kong, Hao Zheng, *The center functor is fully faithful*, Adv. Math. 339 (2018) 749-779 arXiv:1507.00503
- [KZ18b] Liang Kong, Hao Zheng, *Drinfeld center of enriched monoidal categories*, Adv. Math. 323 (2018) 411-426 arXiv:1704.01447
- [KZ18c] Liang Kong, Hao Zheng, *Gapless edges of 2d topological orders and enriched monoidal categories*, Nucl. Phys. B 927 (2018) 140-165 arXiv:1705.01087
- [KZ20] Liang Kong, Hao Zheng, *A mathematical theory of gapless edges of 2d topological orders. Part I*, J. High Energ. Phys. 2020, 150 (2020). arXiv:1905.04924
- [KZ21] Liang Kong, Hao Zheng, *A mathematical theory of gapless edges of 2d topological orders. Part II*, Nucl. Phys. B, 966 (2021), 115384 arXiv:1912.01760
- [KZ22a] Liang Kong, Hao Zheng, *Categories of quantum liquids, I*, J. High Energ. Phys., 2022, 70 (2022) arXiv:2011.02859
- [KZ23] Liang Kong, Hao Zheng, *Categorical computation*, Frontiers of Physics, Volume 18, Issue 2, 21302, 2023 <https://doi.org/10.1007/s11467-022-1251-5>, arXiv:2102.04814
- [KZ24] Liang Kong, Hao Zheng, *Categories of quantum liquids II*, Commun. Math. Phys., 405, 203 (2024) <https://doi.org/10.1007/s00220-024-05078-4>, arXiv:2107.03858
- [KZ22b] Liang Kong, Hao Zheng *Categories of quantum liquids III*, arXiv:2201.05726

- [LKW17a] Tian Lan, Liang Kong, Xiao-Gang Wen, *Modular extensions of unitary braided fusion categories and 2+1D topological/SPT orders with symmetries*, Commun. Math. Phys. 351 (2017) 709-739 [arXiv:1602.05936](#)
- [LKW17b] Tian Lan, Liang Kong, Xiao-Gang Wen, *Classification of 2+1D topological orders and SPT orders for bosonic and fermionic systems with on-site symmetries*, Phys. Rev. B 95, 235140 (2017) [arXiv:1602.05946](#)
- [LKW18] Tian Lan, Liang Kong, and Xiao-Gang Wen. *Classification of 3+1D bosonic topological orders: The case when point-like excitations are all bosons*. Physical Review X, 8(2), 2018. [arXiv:1704.04221](#)
- [LW19] Tian Lan and Xiao-Gang Wen. *Classification of 3+1D bosonic topological orders (II): The case when some pointlike excitations are fermions*. Physical Review X, 9(2), 2019. [arXiv:1801.08530](#)
- [Lev13] Michael Levin. *Protected edge modes without symmetry*. Physical Review X, 3(2), 2013. [arXiv:1301.7355](#)
- [MS89a] Gregory Moore and Nathan Seiberg. *Classical and quantum conformal field theory*. Communications in Mathematical Physics, 123(2):177–254, 1989.
- [MS89b] Gregory Moore and Nathan Seiberg. *Naturality in conformal field theory*, Nuclear Physics B, 313(1):16–40, 1989.
- [MS89c] Gregory Moore and Nathan Seiberg. *Taming the conformal zoo*. Physics Letters B, 220(3):422–430, 1989.
- [MP19] Scott Morrison and David Penneys. *Monoidal categories enriched in braided monoidal categories*. International Mathematics Research Notices, 2019(11):3527–3579, 2017. [arXiv:1701.00567](#)
- [TW19] Ryan Thorngren, Yifan Wang, *Fusion Category Symmetry I: Anomaly In-Flow and Gapped Phases*, J. High Energ. Phys. 2024, 132 (2024). [https://doi.org/10.1007/JHEP04\(2024\)132](https://doi.org/10.1007/JHEP04(2024)132), [arXiv:1912.02817](#)
- [Wen19] Xiao-Gang Wen, *Choreographed entanglement dances: Topological states of quantum matter*, Science 363, eaal3099 (2019).
- [XZ24] Rongge Xu and Zhi-Hao Zhang, *Categorical descriptions of one-dimensional gapped phases with Abelian onsite symmetries*, Physical Review B, 110(15):155106, 2024. [arXiv:2205.09656](#).
- [ZW15] Bei Zeng and Xiao-Gang Wen. *Gapped quantum liquids and topological order, stochastic local transformations and emergence of unitarity*. Physical Review B, 91(12), 2015. [arXiv:1406.5090](#)
- [ZLZ+23] Jiaheng Zhao, Jia-Qi Lou, Zhi-Hao Zhang, Ling-Yan Hung, Liang Kong, Yin Tian, *String condensations in 3+1D and Lagrangian algebras*, Adv. Theor. Math. Phys., Volume 27, Number 2, 583-622, 2023 <https://dx.doi.org/10.4310/ATMP.2023.v27.n2.a5>, [arXiv:2208.07865](#)
- [Zhe17] Hao Zheng, *Extended TQFT's arising from enriched multi-fusion categories*, [arXiv:1704.05956](#)